

NETWORK ANALYSIS AND TRANSMISSION LINES

LECTURE NOTES

B.TECH

ECE

(II YEAR – ISEM)

**Department of Electronics and Communication
Engineering**



MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

Recognized under 2(f) and 12 (B) of UGC ACT 1956

(Affiliated to JNTUH, Hyderabad, Approved by AICTE - Accredited by NBA & NAAC – ‘A’ Grade - ISO 9001:2015 Certified) Maisammaguda, Dhulapally (Post Via. Kompally), Secunderabad – 500100, Telangana State, India

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

II Year B.Tech. ECE-I Sem

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(R18A0261) NETWORK ANALYSIS & TRANSMISSION LINES

COURSE OBJECTIVES:

This course introduces the basic concepts of transient analysis of the circuits, the basic two-port network parameters, design analysis of the filters and attenuators and their use in the circuit theory, analysis of the locus diagrams, resonance, magnetic circuits. The emphasis of this course is laid on the basic operation of DC machines which includes DC generators and DC motors.

UNIT – I:

Transient Analysis (First and Second Order Circuits): Introduction to transient response and steady state response, Transient response of series –RL, RC RLC Circuits for sinusoidal, square, ramp and pulse excitations, Initial Conditions, Solution using Differential Equations approach and Laplace Transform method,

UNIT – II:

Two Port Networks: Impedance Parameters, Admittance Parameters, Hybrid Parameters, Transmission (ABCD) Parameters, Conversion of one of parameter to another, Conditions for Reciprocity and Symmetry, Interconnection of two port networks in Series, Parallel and Cascaded configurations, Image Parameters, Illustrative problems.

UNIT-III:

Locus diagrams: Resonance and Magnetic Circuits: Locus diagrams – Series and Parallel RL, RC, RLC circuits with variation of various parameters – Resonance-Series and Parallel circuits, Concept of band width and quality factor.

Magnetic Circuits- Faraday's laws of electromagnetic induction, Concept of self and mutual inductance, Dot convention, Coefficient of coupling, Composite magnetic circuits, Analysis of series and parallel magnetic circuits.

UNIT – IV:

Transmission Lines – I: Types, Parameters, Transmission Line Equations, Primary & Secondary Constants, Expressions for Characteristics Impedance, Propagation Constant, Phase and Group Velocities, Infinite Line Concepts, Losslessness/Low Loss Characterization, Distortion – Condition for Distortionlessness and Minimum Attenuation, Illustrative Problems.

UNIT V:

Transmission Lines – II: SC and OC Lines, Input Impedance Relations, Reflection Coefficient, VSWR, $\lambda/4$, $\lambda/2$, $\lambda/8$ Lines – Impedance Transformations, Significance of Z_{\min} and Z_{\max} , Smith Chart – Configuration and Applications, Single Stub Matching, Illustrative Problems.

TEXT BOOKS:

1. Electrical Circuits – A. Chakrabarhty, Dhanipat Rai & Sons.
2. Network Analysis – N.C Jagan and C. Lakhminarayana, BS publications.
3. A Text book of Electrical Technology by B.L Theraja and A.K Theraja, S.Chand publications
4. Basic Concepts of Electrical Engineering – PS Subramanyam, BS Publications.
5. Transmission Lines and Networks – Umesh Sinha, Satya prakashan, 2001, (Tech. India Publications), New Delhi.

REFERENCE BOOKS:

1. Engineering Circuits Analysis – William Hayt and Jack E. Kemmerly, Mc Graw Hill Company, 7th Edition.
2. Basic Electrical Engineering – S.N. Singh PUI.
3. Electrical Circuits – David A. Bell, Oxford Printing Press.
4. Principles of Electrical Engineering by V.K Mehta, Rohit Mehta, S.Chand publications.
5. Electrical Circuit Analysis – K.S. Suresh Kumar, Pearson Education.

COURSE OUTCOMES:

After going through this course the student gets a thorough knowledge on Transient analysis of the circuits, filters, attenuators and the operation of DC machines with which he/she can able to apply the above conceptual things to real world problems and applications

UNIT – I:**Transient Analysis (First and Second Order Circuits):**

- **Introduction to transient response and steady state response**
- **Transient response of series –RL, RC RLC Circuits for sinusoidal, square, ramp and pulse excitations**
- **Initial Conditions**
- **Solution using Differential Equations approach and Laplace Transform method**

Introduction to transient response and steady state response

- In this chapter we shall study transient response of the RL, RC series and RLC circuits with sinusoidal, square, ramp and pulse excitations.
- Transients are present in the circuit, when the circuit is subjected to any changes either by changing source magnitude or while changing any circuit elements, provided circuit consists of any energy storage elements.
- There are 3 circuit elements(1)Resistor (2)Inductor(3)Capacitor
- Inductor and Capacitor are called storage elements.
- Inductor doesn't allow sudden change in current and stores the energy in the form of magnetic field.
- Capacitor doesn't allow sudden change in voltage and stores the energy in the form of electric field.
- When the circuit is having only resistive elements, no transients present in the circuit since resistor allows sudden change in current and voltage and it doesn't store any energy.
- **The total response of the circuit=Transient response +Steady state response.**
- **Transient response** changes with time and gets saturated after some time. It is also called as natural response.
- **Steady state response** doesn't change with the time. It is also called forced response.
- The time taken for the circuit to change from one steady state to another steady state is called **transient time**.
- Under initial conditions inductor behaves like open circuit i.e. $I_L=0$
- Under steady state conditions inductor behaves like short circuit i.e. $V_L=0$
- Under initial conditions Capacitor behaves like short circuit i.e. $V_C=0$
- Under steady state conditions capacitor behaves like open circuit i.e. $I_C=0$

$t=0^-$ indicates immediately before operating switch

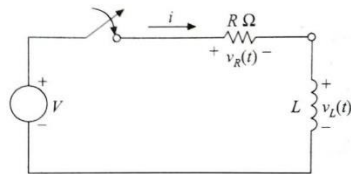


Fig1.1

$t=0^+$ indicates immediately after operating switch

$t=\infty$ indicates steady state condition

$t=0^-$ $i_L=0$

$t=0^+$ $i_L=0$

$t=\infty$ $i_L=V/R$

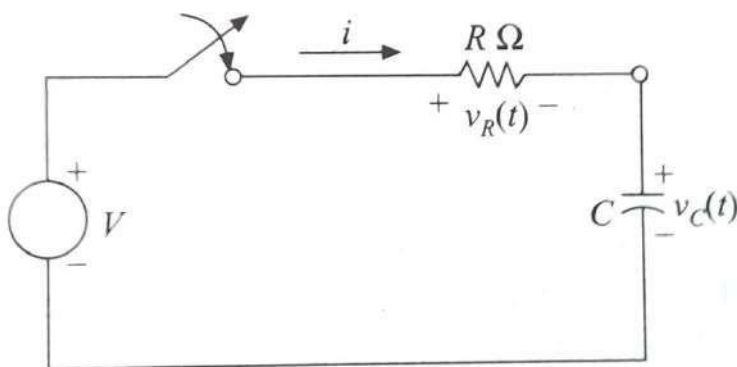


Fig1.2

$t=0^-$ $V_C=0$

$t=0^+$ $V_C=0$

$$t=\infty \quad V_c=V$$

Transient response of series –RL Circuit for sinusoidal excitation

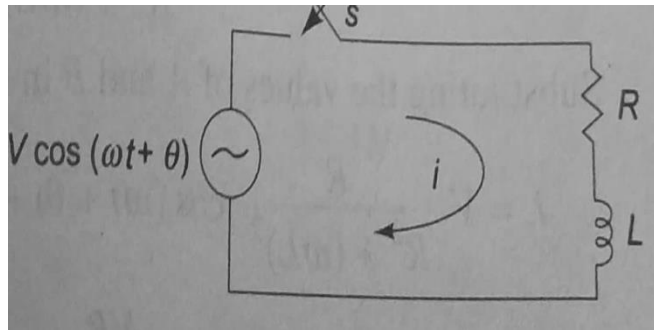


Fig1.3

Consider a circuit consisting of Series resistance and inductance as shown in fig1.3. The switch S is closed at $t=0$.

At $t=0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the series RL circuit, where V is amplitude of the wave and θ is phase angle.

Application of KVL to the circuit results in the following differential equation.

$$V \cos(\omega t + \theta) = Ri + L \frac{di}{dt} \quad (1.1)$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L} \cos(\omega t + \theta)$$

The corresponding characteristic equation is

$$\left(D + \frac{R}{L}\right)i = \frac{V}{L} \cos(\omega t + \theta) \quad (1.2)$$

For the above equation, the solution consists of two parts, viz. complementary function and particular integral.

$$i_c = ce^{-t(R/L)} \quad \text{-----(1.3)}$$

The particular integral can be determined by using undetermined coefficients.

By assuming

$$i_p = A \cos (\omega t + \theta) + B \sin (\omega t + \theta) \quad \text{-----(1.4)}$$

$$i_p' = -A\omega \sin (\omega t + \theta) + B\omega \cos (\omega t + \theta) \quad \text{-----(1.5)}$$

Substituting equations (1.4) and (1.5) in equation (2)

$$\begin{aligned} & \{-A\omega \sin (\omega t + \theta) + B\omega \cos (\omega t + \theta)\} + \frac{R}{L} \{A \cos (\omega t + \theta) \\ & \quad + B \sin (\omega t + \theta)\} = \frac{V}{L} \cos (\omega t + \theta) \\ \text{or } & \left(-A\omega + \frac{BR}{L}\right) \sin (\omega t + \theta) + \left(B\omega + \frac{AR}{L}\right) \cos (\omega t + \theta) = \frac{V}{L} \cos (\omega t + \theta) \end{aligned}$$

Comparing cosine terms and sine terms, we get

$$\begin{aligned} -A\omega + \frac{BR}{L} &= 0 \\ B\omega + \frac{AR}{L} &= \frac{V}{L} \end{aligned}$$

From the above equations, we have

$$\begin{aligned} A &= V \frac{R}{R^2 + (\omega L)^2} \\ B &= V \frac{\omega L}{R^2 + (\omega L)^2} \end{aligned}$$

Substituting the values of A and B in equ(1.4), we get

$$i_p = V \frac{R}{R^2 + (\omega L)^2} \cos(\omega t + \theta) + V \frac{\omega L}{R^2 + (\omega L)^2} \sin(\omega t + \theta)$$

Putting $M \cos \phi = \frac{VR}{R^2 + (\omega L)^2}$

and $M \sin \phi = V \frac{\omega L}{R^2 + (\omega L)^2}$

To find M and ϕ , We divide one equation by the other

$$\frac{M \sin \phi}{M \cos \phi} = \tan \phi = \frac{\omega L}{R}$$

Squaring both equations and adding, we get

$$M^2 \cos^2 \phi + M^2 \sin^2 \phi = \frac{V^2}{R^2 + (\omega L)^2}$$

$$M = \frac{V}{\sqrt{R^2 + (\omega L)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R}\right) \quad \text{----- (1.6)}$$

The complete solution for the current $i = i_c + i_p$

$$i = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R}\right)$$

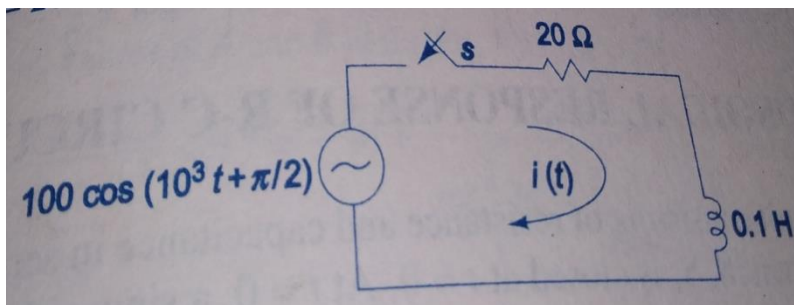
Since the inductor does not allow sudden change in currents, at $t=0$, $i=0$

$$c = - \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1} \frac{\omega L}{R}\right)$$

$$i = e^{-(R/L)t} \left[\frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos \left(\theta - \tan^{-1} \frac{\omega L}{R} \right) \right] + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos \left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R} \right)$$

Example 1.1

In the circuit as shown in figure below, determine the complete solution for the current, when switch S is closed at $t=0$. Applied voltage $v(t)=100\cos(10^3t+\pi/2)$. Resistance $R=20\Omega$ and inductance $L=0.1H$.



Solution

By applying Kirchhoff's voltage law to the circuit, we have

$$20i + 0.1 \frac{di}{dt} = 100 \cos(10^3 t + \pi/2).$$

$$\frac{di}{dt} + 200i = 1000 \cos(1000t + \pi/2)$$

$$(D+200)i = 1000 \cos(1000t + \pi/2)$$

The complementary function $i_c = ce^{-200t}$

By assuming particular integral as

$$i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta)$$

We get

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t + \theta - \tan^{-1} \frac{\omega L}{R})$$

Where $\omega = 1000 \text{ rad/sec}$

$$V=100 \text{ V}, \theta = \frac{\pi}{2}$$

$$L=0.1 \text{ H}, R=20 \Omega$$

Substituting the values in the above equation, we get

$$i_p = \frac{100}{\sqrt{20^2 + (1000 \cdot 0.1)^2}} \cos(1000t + \frac{\pi}{2} - \tan^{-1} \frac{100}{20})$$

$$= \frac{100}{101.9} \cos(1000t + \frac{\pi}{2} - 78.6^\circ)$$

$$= 0.98 \cos(1000t + \frac{\pi}{2} - 78.6^\circ)$$

The complete solution is

$$i = ce^{-200t} + 0.98 \cos(1000t + \frac{\pi}{2} - 78.6^\circ)$$

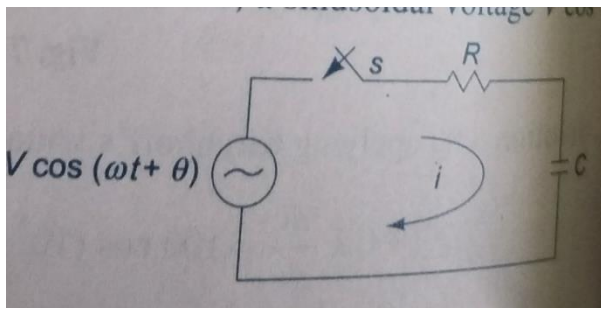
At $t=0$, the current flowing through the circuit is zero, i.e. $i=0$

$$c = -0.98 \cos(\frac{\pi}{2} - 78.6^\circ)$$

The complete solution is

$$i = [-0.98 \cos(\frac{\pi}{2} - 78.6^\circ)] e^{-200t} + 0.98 \cos(1000t + \frac{\pi}{2} - 78.6^\circ)$$

SINUSOIDAL RESPONSE OF R-C CIRCUIT:



Consider a circuit consisting of resistance and capacitance in series as shown in fig. The switch, S, is closed at $t=0$. At $t=0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the R-C circuit, where V is the amplitude of the wave and θ = Phase angle.

Applying KVL to the circuit results in the following differential equation.

$$V \cos(\omega t + \theta) = Ri + \frac{1}{C} \int i dt \text{ ----- (1.7)}$$

$$R \frac{di}{dt} + \frac{i}{C} = -V\omega (\sin \omega t + \theta)$$

$$(D + \frac{1}{RC})i = \frac{-V\omega}{R} (\sin \omega t + \theta) \text{ ----- (1.8)}$$

$$\text{The complementary function } i_c = K e^{-t/RC} \text{ ----- (1.9)}$$

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A \cos(\omega t + \theta) + B \sin \omega t + \theta \text{ ----- (1.10)}$$

$$i_p' = -A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta) \text{ ----- (1.11)}$$

Substituting equations 1.10 and 1.11 in 1.8 we get

$$\{-A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta)\} + \frac{1}{RC} A \cos(\omega t + \theta) + B \sin \omega t + \theta = -\frac{V\omega \sin \omega t + \theta}{R}$$

Comparing both sides

$$-A\omega + \frac{B}{RC} = \frac{-V\omega}{R}$$

$$B\omega + \frac{A}{RC} = 0$$

From which,

$$A = \frac{VR}{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$B = -\frac{V}{\omega C \left(R^2 + \left(\frac{1}{\omega C} \right)^2 \right)}$$

Substituting values of A and B in equation (1.10), we have

$$i_p = \frac{VR}{R^2 + \left(\frac{1}{\omega C} \right)^2} \cos(\omega t + \theta) - \frac{V}{\omega C \left(R^2 + \left(\frac{1}{\omega C} \right)^2 \right)} \sin \omega t + \theta$$

Putting

$$M \cos \phi = \frac{VR}{R^2 + \left(\frac{1}{\omega C} \right)^2}$$

$$M \sin \phi = \frac{V}{\omega C \left(R^2 + \left(\frac{1}{\omega C} \right)^2 \right)}$$

To find out M and ϕ , we divide one equation by other,

$$\frac{M \cos \phi}{M \sin \phi} = \tan \phi = \frac{1}{\omega CR}$$

Squaring both sides and adding, we get

$$(M \cos \phi)^2 + (M \sin \phi)^2 = \frac{V^2}{\left(R^2 + \left(\frac{1}{\omega C} \right)^2 \right)}$$

$$M = \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} \right)^2 \right)}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} \right)^2 \right)}} \cos \left(\omega t + \theta + \tan^{-1} \frac{1}{\omega CR} \right) \text{----- (1.12)}$$

The complete solution for the current $i=i_c+i_p$

$$i = K e^{-t/RC} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1} \frac{1}{\omega CR}\right) \text{----- (1.13)}$$

Since the capacitor does not allow sudden change in voltages at $t=0$, $i = \frac{V}{R} \cos \theta$

$$\frac{V}{R} \cos \theta = K + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1} \frac{1}{\omega CR}\right)$$

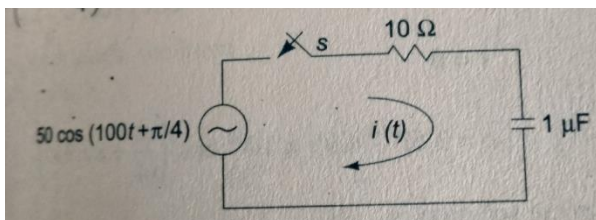
$$K = \frac{V}{R} \cos \theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1} \frac{1}{\omega CR}\right)$$

The complete solution for the current is

$$I = e^{-\frac{t}{RC}} \left[\frac{V}{R} \cos \theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1} \frac{1}{\omega CR}\right) \right] + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1} \frac{1}{\omega CR}\right) \text{----- (1.14)}$$

Example 1.2.

In the circuit as shown in Figure below, determine the complete solution for the current when switch S is closed at $t=0$. Applied voltage is $v(t)=50\cos(10^2t+\pi/4)$. Resistance $R=10\Omega$ and capacitance $C=1\mu F$.



Solution:

By applying KVL to the circuit, we have

$$10i + \frac{1}{10^{-6}} \int i dt = 50 \cos(100t + \pi/4)$$

$$10 \frac{di}{dt} + \frac{i}{10^{-6}} = -5 \times 10^3 (\sin 100t + \pi/4)$$

$$\frac{di}{dt} + \frac{i}{10^{-5}} = 500 (\sin 100t + \pi/4)$$

$$(D + \frac{1}{10^{-5}})i = -500 (\sin 100t + \pi/4)$$

The complementary function $i_c = K e^{-t/10^{-5}}$

The particular solution $i_p = A \cos(\omega t + \theta) + B \sin \omega t + \theta$

$$\text{We get } i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1} \frac{1}{\omega CR}\right)$$

$$\text{Where } \omega = 100 \frac{\text{rad}}{\text{sec}} \quad \theta = \frac{\pi}{4}$$

$$R = 10 \Omega \quad C = 1 \mu F$$

$$i_p = \frac{500}{\sqrt{10^2 + \left(\frac{1}{100 \times 10^{-6} \times 10}\right)^2}} \cos\left(100t + \frac{\pi}{4} + \tan^{-1} \frac{1}{100 \times 10^{-6} \times 10}\right)$$

$$i_p = 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

$$\text{at } t = 0, i = \frac{V}{R} \cos \theta = \frac{50}{10} \cos \frac{\pi}{4} = 3.53 A$$

$$i = K e^{-t/10^{-5}} + 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

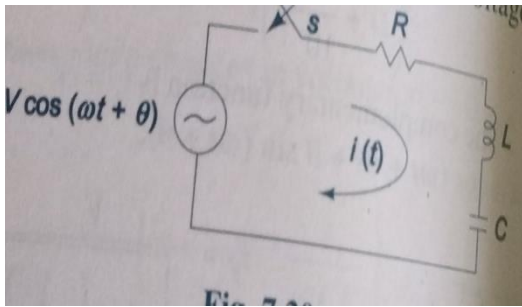
At $t=0$

$$K = 3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94^\circ\right)$$

Hence the complete solution is

$$i = \left[3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94^\circ\right) \right] e^{-t/10^{-5}} + 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

SINUSOIDAL RESPONSE OF RLC CIRCUIT:



Consider a circuit consisting of resistance, inductance and capacitance in series as shown in fig. The switch, S is closed at $t=0$. At $t=0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the RLC series circuit, where V is the amplitude of the wave and θ = Phase angle.

Applying KVL to the circuit results in the following differential equation.

$$V \cos(\omega t + \theta) = RI + L \frac{di}{dt} + \frac{1}{C} \int i dt \text{ ----- (1.15)}$$

Differentiating above equation, we get

$$R \frac{di}{dt} + L \frac{d^2 i}{dt^2} + \frac{i}{C} = -V \omega \sin(\omega t + \theta)$$

$$\left(D^2 + \frac{R}{L} D + \frac{1}{LC} \right) i = -\frac{V \omega}{L} \sin(\omega t + \theta) \text{ ----- (1.16)}$$

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta) \text{ ----- (1.17)}$$

$$i_p^1 = -A \omega \sin(\omega t + \theta) + B \omega \cos(\omega t + \theta) \text{ ----- (1.18)}$$

$$i_p'' = -A\omega^2 \cos(\omega t + \theta) - B\omega^2 \sin(\omega t + \theta) \text{ ----- (1.19)}$$

Substituting values of i_p , i_p' , i_p'' in equ (1.16) we have

$$-A\omega^2 \cos(\omega t + \theta) - B\omega^2 \sin(\omega t + \theta) + \frac{R}{L} [-A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta)] + \frac{1}{LC} [A \cos(\omega t + \theta) + B \sin(\omega t + \theta)] = -\frac{V\omega}{L} \sin(\omega t + \theta) \text{ ----- (1.20)}$$

Comparing both sides, we have

Sine coefficients

$$-B\omega^2 - A\omega \frac{R}{L} + \frac{B}{LC} = -\frac{V\omega}{L}$$

$$A\omega \frac{R}{L} + B \left(\omega^2 - \frac{1}{LC} \right) = \frac{V\omega}{L} \text{ ----- (1.21)}$$

Cosine coefficients

$$-A\omega^2 + B\omega \frac{R}{L} + \frac{A}{LC} = 0$$

$$A \left(\omega^2 - \frac{1}{LC} \right) - B \left(\frac{\omega R}{L} \right) = 0 \text{ ----- (1.22)}$$

Solving (1.21) and (1.22) we get

$$A = \frac{\frac{V\omega^2 R}{L^2}}{\left[\left(\frac{\omega R}{L} \right)^2 - \left(\left(\omega^2 - \frac{1}{LC} \right)^2 \right) \right]}$$

$$B = \frac{\left(\omega^2 - \frac{1}{LC} \right)^2 V\omega}{L \left[\left(\frac{\omega R}{L} \right)^2 - \left(\left(\omega^2 - \frac{1}{LC} \right)^2 \right) \right]}$$

Substituting values of A and B in equation (1.17), We get

$$i_p = \frac{\frac{V\omega^2 R}{L^2}}{\left[\left(\frac{\omega R}{L} \right)^2 - \left(\left(\omega^2 - \frac{1}{LC} \right)^2 \right) \right]} \cos(\omega t + \theta) + \frac{\left(\omega^2 - \frac{1}{LC} \right)^2 V\omega}{L \left[\left(\frac{\omega R}{L} \right)^2 - \left(\left(\omega^2 - \frac{1}{LC} \right)^2 \right) \right]} \sin(\omega t + \theta) \text{ ----- (1.23)}$$

Putting

$$M \cos \phi = \frac{\frac{V \omega^2 R}{L^2}}{\left[\left(\frac{\omega R}{L} \right)^2 - \left(\omega^2 - \frac{1}{LC} \right)^2 \right]}$$

$$M \sin \phi = \frac{\left(\omega^2 - \frac{1}{LC} \right)^2 V \omega}{L \left[\left(\frac{\omega R}{L} \right)^2 - \left(\omega^2 - \frac{1}{LC} \right)^2 \right]}$$

To find out M and ϕ , we divide one equation by other,

$$\frac{M \cos \phi}{M \sin \phi} = \tan \phi = \frac{\left(\omega L - \frac{1}{\omega C} \right)}{R}$$

$$\phi = \tan^{-1} \left[\frac{\left(\omega L - \frac{1}{\omega C} \right)}{R} \right]$$

Squaring both equations and adding we get

$$(M \cos \phi)^2 + (M \sin \phi)^2 = \frac{V^2}{\left(R^2 + \left(\frac{1}{\omega C} - \omega L \right)^2 \right)}$$

$$M = \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} - \omega L \right)^2 \right)}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} - \omega L \right)^2 \right)}} \cos \left(\omega t + \theta + \tan^{-1} \left[\frac{\left(\omega L - \frac{1}{\omega C} \right)}{R} \right] \right) \text{-----(1.24)}$$

To find out complementary function, we have the characteristic equation

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0 \text{----- (1.25)}$$

The roots of equation(1.25) are

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $K_1 = -\frac{R}{2L}$

$$K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$D_1 = K_1 + K_2$$

$$D_1 = K_1 - K_2$$

K_2 becomes positive, when $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

The roots are real and unequal, which gives an over damped response. Then equation (1.25) becomes

$$[D - (K_1 + K_2)][D - (K_1 - K_2)]i = 0$$

The complementary function of above equation is

$$i_c = c_1 e^{(k_1 + k_2)t} + c_2 e^{(k_1 - k_2)t} + \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right)}} \cos\left(\omega t + \theta + \tan^{-1}\left[\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}\right]\right)$$

K_2 becomes negative when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

Then the roots are complex conjugate, which gives an under damped response.

Then equation (1.25) becomes

$$[D - (K_1 + jK_2)][D - (K_1 - jK_2)]i = 0$$

The solution for above equation is

$$i_c = e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t]$$

$$i = i_c + i_p$$

$$i = e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] + \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right)}} \cos \left(\omega t + \theta + \tan^{-1} \left[\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R} \right] \right)$$

$$k_2 \text{ becomes zero when } \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

Then the roots are equal which gives critically damped response

$$\text{Then equation (1.25) becomes } (D - K_1)(D - K_1)i = 0$$

The complementary function for the above equation is

$$i_c = e^{(k_1)t} [c_1 + c_2 t]$$

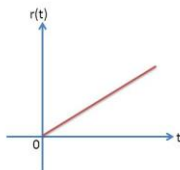
Therefore complete solution is $i = i_c + i_p$

$$e^{(k_1)t} [c_1 + c_2 t] + \frac{V}{\sqrt{\left(R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right)}} \cos \left(\omega t + \theta + \tan^{-1} \left[\frac{\left(\frac{1}{\omega C} - \omega L\right)}{R} \right] \right)$$

$$F(s) = \mathcal{L} \{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Solution using Laplace transformation method:

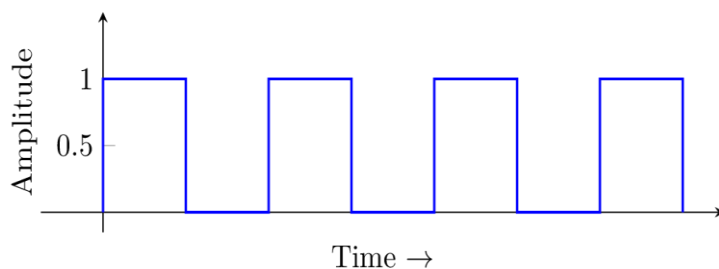
$f(t)$ (Function)	$F(s)$ (Laplace Transform)
$u(t)$ (unit step)	$1/s$
$\delta(t)$ (unit impulse)	1
e^{-at}	$\frac{1}{(s+a)}$
$\sin \omega t$	$\frac{\omega}{(s^2 + \omega^2)}$
$\cos \omega t$	$\frac{s}{(s^2 + \omega^2)}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2 + \omega^2}$
t	$1/s^2$
$\frac{df(t)}{dt}$	$sF(s)$
$\int f(t)dt$	$F(s)/s$

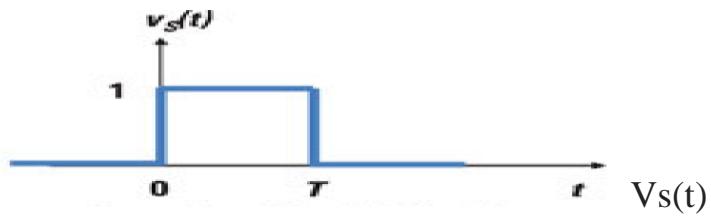
Ramp input**UNIT RAMP INPUT**

- The ramp signal imitates the constant velocity characteristic of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

If $A=1$, the ramp signal is called unit ramp signal

Square input**Square wave**

Pulse input

$$V_s(t) = u(t) - u(t-T)$$

$$V_s(s) = \frac{1 - e^{-sT}}{s}$$

UNIT – II:**Two Port Networks:**

- Impedance Parameters,
- Admittance Parameters,
- Hybrid Parameters,
- Transmission (ABCD) Parameters,
- Conversion of one of parameter to another,
- Conditions for Reciprocity and Symmetry,
- Interconnection of two port networks in Series, Parallel and Cascaded configurations, Image Parameters,
- Illustrative problems.

Introduction:

A general network having two pairs of terminals, one labeled the “input terminals” and the other the “output terminals,” is a very important building block in electronic systems, communication systems, automatic control systems, transmission and distribution systems, or other systems in which an electrical signal or electric energy enters the input terminals, is acted upon by the network, and leaves via the output terminals. A pair of terminals at which a signal may enter or leave a network is also called a **port**, and a network like the above having two such pair of terminals is called a **Two - port network**. A general two-port network with terminal voltages and currents specified is shown in the figure below. In such networks the relation between the two voltages and the two currents can be described in six different ways resulting in six different systems of Parameters and in this chapter we will consider the most important four systems

Impedance Parameters: Z parameters (open circuit impedance parameters)

We will assume that the two port networks that we will consider are composed of linear elements and contain no independent sources but dependent sources *are* permissible. We will consider the two-port network as shown in the figure below.

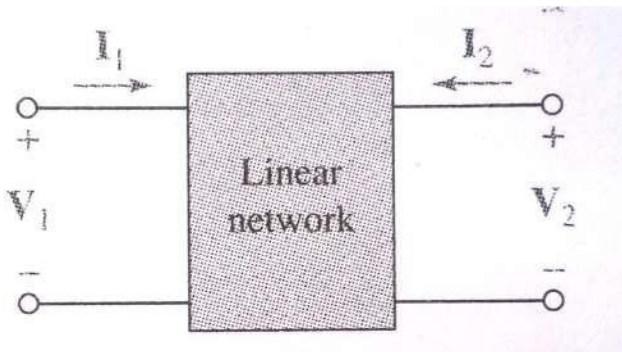


Fig 5.1: A general two-port network with terminal voltages and currents specified. The two- port network is composed of linear elements, possibly including dependent sources, but not containing any independent sources.

The voltage and current at the input terminals are V_1 & I_1 , and V_2 & I_2 are voltage and current at the output port. The directions of I_1 and I_2 are both customarily selected as into the network at the upper conductors (and out at the lower conductors). Since the network is

Linear and contains no independent sources within it, V_1 may be considered to be the superposition of two components, one caused by I_1 and the other by I_2 . When the same argument is applied to V_2 , we get the set of equations

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

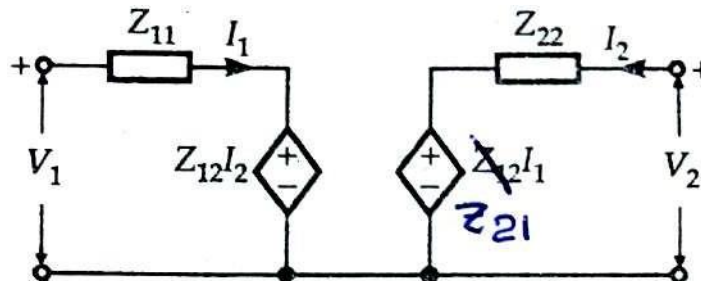
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$[V] = [Z][I]$$

Where $[V]$, $[Z]$ and $[I]$ are Voltage, impedance and current matrices. The description of the Z parameters, defined in the above equations is obtained by setting each of the currents equal to zero as given below.

$$Z_{11} = V_1/I_1 \mid I_2=0 \quad Z_{12} = V_1/I_2 \mid I_1=0 \quad Z_{21} = V_2/I_1 \mid I_2=0 \quad Z_{22} = V_2/I_2 \mid I_1=0$$

Thus, since zero current results from an open-circuit termination, the Z parameters are known as the **Open-circuit Impedance parameters**. And more specifically Z_{11} & Z_{22} are called **Driving point Impedances** and Z_{12} & Z_{21} are called **Reverse and Forward transfer impedances** respectively. A basic Z parameter equivalent circuit depicting the above defining equations is



shown in the figure below.

Fig 5.2: Z-Parameter equivalent circuit

Admittance parameters: (Y Parameters or Short circuit admittance parameters)

The same general two port network shown for Z parameters is applicable here also and is shown below.

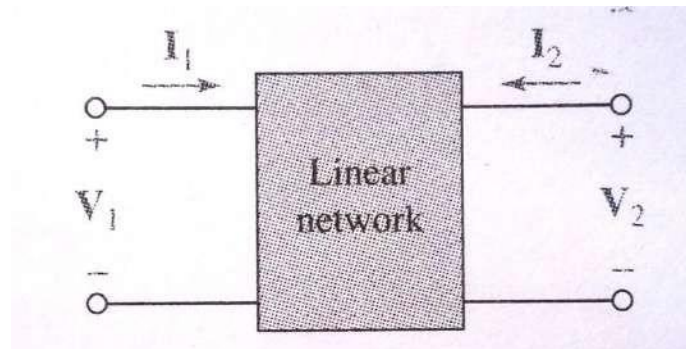


Fig 5.3: A general two-port network with terminal voltages and currents specified. The two- port network is composed of linear elements, possibly including dependent sources, but not containing any independent sources.

Since the network is linear and contains no independent sources within, on the same lines of Z parameters the defining equations for the Y parameters are given below. I_1 and I_2 may be considered to be the superposition of two components, one caused by V_1 and the other by V_2 and then we get the set of equations defining the Y parameters.

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Where the Y s are no more than proportionality constants and their dimensions are A/V (Current/Voltage). Hence they are called the Y (or admittance) parameters. They are also defined in the matrix form given below.

And in much simpler form as

$$[\mathbf{I}] = [\mathbf{Y}][\mathbf{V}]$$

The individual Y parameters are defined on the same lines as Z parameters but by setting either of the voltages V_1 and V_2 as zero as given below.

The most informative way to attach a physical meaning to the y parameters is through a direct inspection of defining equations. The conditions which must be applied to the basic defining equations are very important. In the first equation for example; if we let V_2 zero, then Y_{11} is given by the ratio of I_1 to V_1 . We therefore describe Y_{11} as the admittance measured at the input terminals with the output terminals *short-circuited* ($V_2 = 0$). Each of the Y parameters may be described as a **current-voltage** ratio with either $V_1 = 0$ (the input terminals short circuited) or $V_2 = 0$ (the output terminals short-circuited):

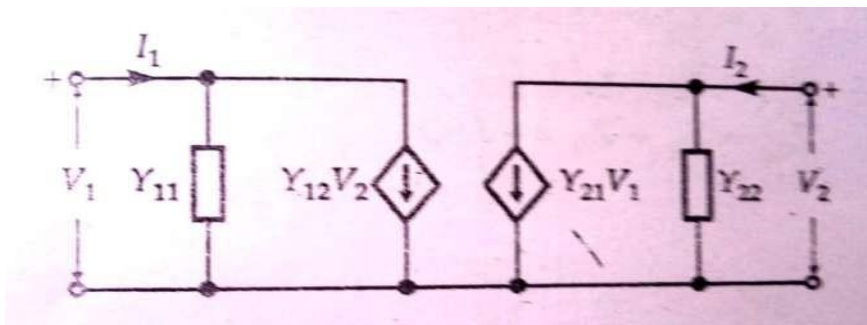
$$Y_{11} = I_1/V_1 \quad \text{with} \quad V_2 = 0$$

$$Y_{12} = I_1/V_2 \quad \text{with} \quad V_1 = 0$$

$$Y_{21} = I_2/V_1 \quad \text{with} \quad V_2 = 0$$

$$Y_{22} = I_2/V_2 \quad \text{with} \quad V_1 = 0$$

Because each parameter is an **admittance** which is obtained by short circuiting either the output or the input port, the Y parameters are known as the **short-circuit admittance parameters**. The specific name of Y_{11} is the **short-circuit input admittance**, Y_{22} is the **short circuit output admittance**, and Y_{12} and Y_{21} are the **short-circuit reverse and forward transfer admittances respectively**.



h parameter representation is used widely in modeling of Electronic components and circuits particularly Transistors. Here both short circuit and open circuit conditions are utilized.

The hybrid parameters are defined by writing the pair of equations relating V_1 , I_1 , V_2 , and I_2 :

$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$$

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$$

The nature of the parameters is made clear by first setting $V_2 = 0$. Thus,

$$h_{11} = V_1/I_1 \text{ with } V_2=0 = \text{short-circuit input impedance}$$

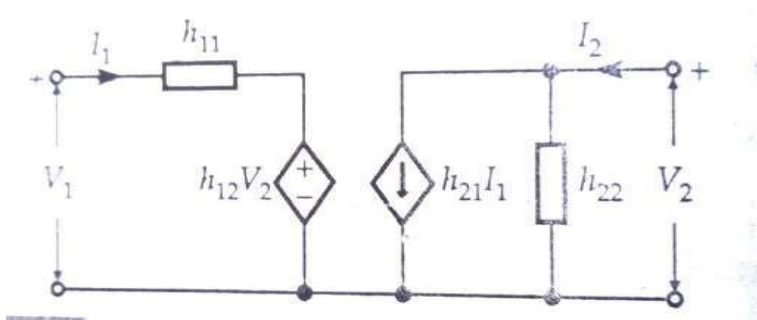
$$h_{21} = I_2/I_1 \text{ with } V_2=0 = \text{short-circuit forward current gain}$$

Then letting $I_1 = 0$, we obtain $h_{12} = V_1/V_2$ with $I_1=0 = \text{open-circuit reverse voltage gain}$

$$h_{22} = I_2/V_2 \text{ with } I_1=0 = \text{open-circuit output admittance}$$

Since the parameters represent an impedance, an admittance, a voltage gain, and a current gain, they are called the “hybrid” parameters.

The subscript designations for these parameters are often simplified when they are applied to transistors. Thus, h_{11} , h_{12} , h_{21} , and h_{22} become h_i , h_r , h_f , and h_o , respectively, where the subscripts denote input, reverse, forward, and output.



Transmission parameters:

The last two-port parameters that we will consider are called the **t parameters**, the **ABCD parameters**, or simply the **transmission parameters**. They are defined by the equations

$$V_1 = A.V_2 - B.I_2$$

$$I_1 = C.V_2 - D.I_2$$

and in Matrix notation these equations can be written in the form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where V_1 , V_2 , I_1 , and I_2 are defined as shown in the figure below.

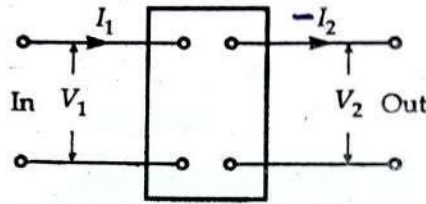


Fig 5.6: Two port Network for ABCD parameter representation with Input and output Voltages and currents

The minus signs that appear in the above equations should be associated with the output current, as $(-I_2)$. Thus, both I_1 and $-I_2$ are directed to the right, the direction of energy or signal transmission.

Note that there are no minus signs in the **t** or **ABCD** matrices. Looking again at the above equations

we see that the quantities on the left, often thought of as the given or independent variables, are the input voltage and current, V_1 and I_1 ; the dependent variables, V_2 and I_2 , are the output quantities. Thus, the transmission parameters provide a direct relationship between input and output. Their major use arises in transmission-line analysis and in cascaded networks.

The four Transmission parameters are defined and explained below.

$$A = V_1/V_2 \text{ with } I_2 = 0 = \text{Reverse voltage Ratio C}$$

$$= I_1/V_2 \text{ with } I_2 = 0 = \text{Transfer admittance}$$

Next **B** and **D** are defined with receiving end short circuited i.e. with $V_2 = 0$

$$B = V_1/-I_2 \text{ with } V_2 = 0 = \text{Transfer impedance}$$

$$D = I_1/-I_2 \text{ with } V_2 = 0 = \text{Reverse current ratio}$$

Inter relationships between different parameters of two port networks:

Basic Procedure for representing any of the above four two port Network parameters in terms of the other parameters consists of the following steps:

1. Write down the defining equations corresponding to the parameters in terms of which the other parameters are to be represented.
2. Keeping the basic parameters same, rewrite/manipulate these two equations in such a way that the variables V_1, V_2, I_1 , and I_2 are arranged corresponding to the defining equations of the first parameters.
3. Then by comparing the parameter coefficients of the respective variables V_1, V_2, I_1 , and I_2 on the right hand side of the two sets of equations we can get the interrelationship.

Z Parameters in terms of Y parameters:

Though this relationship can be obtained by the above steps, the following simpler method is used for **Z in terms of Y** and **Y in terms of Z**:

Z and Y being the Impedance and admittance parameters (Inverse), in matrix notation they are governed by the following inverse relationship.

$$[Z] = [Y]^{-1}$$

Or:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1}$$

Thus :

$$Z_{11} = \frac{Y_{22}}{\Delta Y}, Z_{12} = -\frac{Y_{12}}{\Delta Y}$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} \text{ and } Z_{22} = \frac{Y_{11}}{\Delta Y}$$

$$\left[\text{Here } \Delta Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = Y_{11} Y_{22} - Y_{12} Y_{21} \right]$$

$$V_2 = \frac{1}{C} \cdot I_1 + \frac{D}{C} \cdot I_2$$

Z Parameters in terms of ABCD parameters:

The governing equations are:

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

from the second governing equation $[I_1 = CV_2 - DI_2]$ we can write

$$V_1 = \left[\frac{1}{C} \cdot I_1 + \frac{D}{C} \cdot I_2 \right] A - BI_2$$

$$= \frac{A}{C} \cdot I_1 + \frac{AD - BC}{C} \cdot I_2$$

$$Z_{11} = \frac{A}{C}, Z_{12} = \frac{AD - BC}{C}$$

$$Z_{21} = \frac{1}{C}, Z_{22} = \frac{D}{C}$$

Now substituting this value of V_2 in the first governing equation $[V_1 = AV_2 - BI_2]$ we get

Comparing these two equations for V_1 and V_2 with the governing equations of the **Z** parameter network we get **Z** Parameters in terms of ABCD parameters:

Z Parameters in terms of h parameters:

The governing equations of h parameter network are: $V_1 = h_{11}I_1 + h_{12}V_2$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

From the second equation we get

$$V_2 = -\frac{h_{21}}{h_{22}} \cdot I_1 + \frac{1}{h_{22}} \cdot I_2$$

Substituting this value of V_2 in the first equation for V_1 we get:

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ &= h_{11}I_1 + h_{12} \left[-\frac{h_{21}}{h_{22}}I_1 + \frac{1}{h_{22}}I_2 \right] \\ &= \frac{\Delta h}{h_{22}}I_1 + \frac{h_{12}}{h_{22}}I_2 \end{aligned}$$

Now comparing these two equations for V_1 and V_2 with the governing equations of the **Z** Parameter network we get **Z** Parameters in terms of **h** parameters:

$$\begin{aligned} Z_{11} &= \frac{\Delta h}{h_{22}}, & Z_{12} &= \frac{h_{12}}{h_{22}} \\ Z_{21} &= -\frac{h_{21}}{h_{22}}, & Z_{22} &= \frac{1}{h_{22}} \end{aligned}$$

$$\text{Here } \Delta h = h_{11}h_{22} - h_{12}h_{21}$$

Y Parameters in terms of Z parameters:

Y and Z being the admittance and Impedance parameters (Inverse), in matrix notation they are governed by the following inverse relationship.

$$[Y] = [Z]^{-1}$$

Or:

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1}$$

Thus:

$$Y_{11} = \frac{Z_{22}}{\Delta Z}, \quad Y_{12} = -\frac{Z_{12}}{\Delta Z}$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z}, \quad Y_{22} = \frac{Z_{11}}{\Delta Z}$$

$$\text{Here } \Delta Z = Z_{11} Z_{22} - Z_{12} Z_{21}$$

The other inter relationships also can be obtained on the same lines following the *basic three steps given* in the beginning.

Conditions for reciprocity and symmetry in two port networks:

A two port network is said to be **reciprocal** if the ratio of the output response variable to the input excitation variable is same when the excitation and response ports are interchanged.

A two port network is said to be **symmetrical** if the port voltages and currents remain the same when the input and output ports are interchanged.

In this topic we will get the conditions for **Reciprocity** and **symmetry** for all the four networks. The basic procedure for each of the networks consists of the following steps:

Reciprocity:

- First we will get an expression for the ratio of response to the excitation in terms of the **particular parameters** by giving voltage as excitation at the input port and considering the current in the output port as response (by short circuiting the output port i.e. setting V_2 as zero). i.e. find out (I_2 / V_1)
- Then we will get an expression for the ratio of response to the excitation in terms of the **same parameters** by giving voltage as excitation at the output port and considering the current in the input port as response (by short circuiting the input port i.e. setting V_1 as zero). i.e. find out (I_1 / V_2)
- Equating the RHS of these two expressions would be the condition for reciprocity

Symmetry:

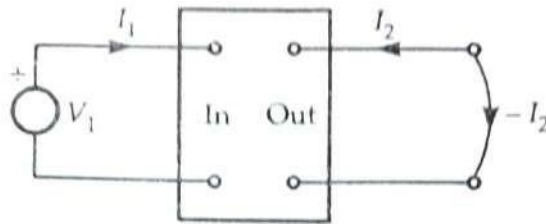
- First we need to get expressions related to the input and output ports using the basic Z or Y parameter equations.
- Then the expressions for Z_{11} and Z_{22} (or Y_{11} and Y_{22}) are equated to get the condition for reciprocity.

Z parameter representation: Condition for reciprocity:

Let us take a two port network with **Z** parameter defining equations as given below:

$$\begin{aligned} V_1 &= Z_{11}I_1 + Z_{12}I_2 \\ V_2 &= Z_{21}I_1 + Z_{22}I_2 \end{aligned}$$

First we will get an expression for the ratio of response (I_2) to the excitation (V_1) in terms of the **Z parameters** by giving excitation at the input port and considering the current in the output port as response (by short circuiting the output port i.e. setting V_2 as zero). The corresponding **Z** parameter circuit for this condition is shown in the figure below:



(Please note the direction of I_2 is negative since when V_2 port is shorted the current flows in the other direction)

Then the **Z** parameter defining equations are :

$$\begin{aligned} V_1 &= Z_{11} \cdot I_1 - Z_{12} \cdot I_2 \text{ and } 0 \\ &= Z_{21} \cdot I_1 - Z_{22} \cdot I_2 \end{aligned}$$

*To get the ratio of response (I_2) to the excitation (V_1) in terms of the **Z** parameters I_1 is to be eliminated from the above equations.*

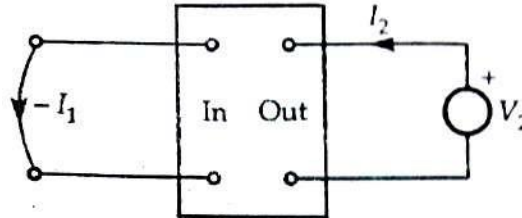
So from equation 2 in the above set we will get $I_1 = I_2 \cdot Z_{22} / Z_{21}$

And substitute this in the first equation to get

$$\begin{aligned} V_1 &= (Z_{11} \cdot I_2 \cdot Z_{22} / Z_{21}) - Z_{12} \cdot I_2 = I_2 [(Z_{11} \cdot Z_{22} / Z_{21}) - Z_{12}] = I_2 [(Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21}) / Z_{21}] \\ I_2 &= V_1 \cdot Z_{21} / (Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21}) \end{aligned}$$

Next, we will get an expression for the ratio of response (I_1) to the excitation (V_2) in terms of the **Z**

parameters by giving excitation V_2 at the output port and considering the current I_1 in the input port as response (by short circuiting the input port i.e. setting V_1 as zero). The corresponding **Z** parameter circuit for this condition is shown in the figure below:



(Please note the direction of current I_1 is negative since when V_1 port is shorted the current flows in the other direction)

Then the Z parameter defining equations are :

$$0 = -Z_{11} \cdot I_1 + Z_{12} \cdot I_2 \text{ and}$$

$$V_2 = -Z_{21} \cdot I_1 + Z_{22} \cdot I_2$$

To get the ratio of response (I_1) to the excitation (V_2) in terms of the Z parameters I_2 is to be eliminated from the above equations.

So from equation 1 in the above set we will get $I_2 = I_1 \cdot Z_{11} / Z_{12}$

And substitute this in the second equation to get

$$V_2 = (Z_{22} \cdot I_1 \cdot Z_{11} / Z_{12}) - Z_{21} \cdot I_1 = I_1 [(Z_{11} \cdot Z_{22} / Z_{12}) - Z_{21}] = I_1 [(Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21}) / Z_{12}]$$

$$I_1 = V_2 \cdot Z_{12} / (Z_{11} \cdot Z_{22} - Z_{12} \cdot Z_{21})$$

Assuming the input excitations V_1 and V_2 to be the same, then the condition for the out responses I_1 and I_2 to be equal would be

$$Z_{12} = Z_{21}$$

And this is the condition for the reciprocity.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports using the basic Z parameter equations.

$$\begin{aligned} V_1 &= Z_{11}I_1 + Z_{12}I_2 \\ V_2 &= Z_{21}I_1 + Z_{22}I_2 \end{aligned}$$

To get the input port impedance I_2 is to be made zero. i.e V_2 should be open.

$$V_1 = Z_{11} \cdot I_1 \quad \text{i.e} \quad Z_{11} = V_1/I_1 \mid I_2=0$$

Similarly to get the output port impedance I_1 is to be made zero. i.e V_1 should be open.

$$V_2 = Z_{22} \cdot I_2 \quad \text{i.e} \quad Z_{22} = V_2/I_2 \mid I_1=0$$

Condition for Symmetry is obtained when the two port voltages are equal i.e. $V_1 = V_2$ and the two port currents are equal i.e. $I_1 = I_2$. Then

$$V_1/I_1 = V_2/I_2 \quad \text{i.e} \quad Z_{11} = Z_{22}$$

And hence $Z_{11} = Z_{22}$ is the condition for symmetry in Z parameters .

Y parameter representation:**Condition for reciprocity :**

Let us take a two port network with Y parameter defining equations as given below:

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

First we will get an expression for the ratio of response (I_2) to the excitation (V_1) in terms of the Y parameters by giving excitation (V_1) at the input port and considering the current (I_2) in the output port as response (by short circuiting the output port i.e. setting V_2 as zero)

Then the second equation in Y parameter defining equations would become

$$I_2 = Y_{21}V_1 + 0 \text{ and } I_2 / V_1 = Y_{21}$$

Then we will get an expression for the ratio of response (I_1) to the excitation (V_2) in terms of the Y parameters by giving excitation (V_2) at the output port and considering the current (I_1) in the input port as response (by short circuiting the input port i.e. setting V_1 as zero)

Then the first equation in Y parameter defining equations would become

$$I_1 = 0 + Y_{12}V_2 \text{ and } I_1 / V_2 = Y_{12}$$

Assuming the input excitations V_1 and V_2 to be the same, then the condition for the out responses I_1 and I_2 to be equal would be

$$I_1 / V_2 = I_2 / V_1$$

And hence $Y_{12} = Y_{21}$ is the condition for the reciprocity in the Two port network with Y parameter representation.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports (In this case Input and output admittances) using the basic Y parameter equations

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

To get the input port admittance, V_2 is to be made zero. i.e V_2 should be shorted.

$$I_1 = Y_{11} \cdot V_1 \quad \text{i.e} \quad Y_{11} = I_1/V_1 \mid V_2=0$$

Similarly to get the output port admittance V_1 is to be made zero. i.e V_1 should be shorted.

$$I_2 = Y_{22} \cdot V_2 \quad \text{i.e} \quad Y_{22} = I_2/V_2 \mid V_1=0$$

Condition for Symmetry is obtained when the two port voltages are equal i.e. $V_1 = V_2$ and the two port currents are equal i.e. $I_1 = I_2$. Then

$$I_1/V_1 = I_2/V_2$$

And hence $Y_{11} = Y_{22}$ is the condition for symmetry in Y parameters.

ABCD parameter representation:**Condition for reciprocity:**

Let us take a two port network with ABCD parameter defining equations as given below:

$$V_1 = A.V_2 - B.I_2$$

$$I_1 = C.V_2 - D.I_2$$

First we will get an expression for the ratio of response (I_2) to the excitation (V_1) in terms of the **ABCD parameters** by giving excitation (V_1) at the input port and considering the current (I_2) in the output port as response (by short circuiting the output port i.e. setting V_2 as zero)

Then the first equation in the **ABCD** parameter defining equations would become

$$V_1 = 0 - B.I_2 = -B.I_2$$

$$\text{i.e } I_2 / V_1 = -1/B$$

Then we will interchange the excitation and response i.e. we will get an expression for the ratio of response (I_1) to the excitation (V_2) by giving excitation (V_2) at the output port and considering the current (I_1) in the input port as response (by short circuiting the input port i.e. setting V_1 as zero)

Then the above defining equations would become

$$0 = A.V_2 - B.I_1$$

$$= C.V_2 - D.I_2$$

Substituting the value of $I_2 = A.V_2 / B$ from first equation into the second equation we get

$$I_1 = C.V_2 - D.A.V_2 / B = V_2 (C - D.A / B)$$

$$\text{i.e } I_1 / V_2 = (BC - DA) / B = -(AD - BC) / B$$

Assuming the input excitations V_1 and V_2 to be the same, then the condition for the out responses I_1 and I_2 to be equal would be

$$I_1 / V_2 = I_2 / V_1$$

$$\text{i.e } -(AD - BC) / B = -1/B$$

$$\text{i.e } (AD - BC) = 1$$

And hence $AD - BC = 1$ is the condition for Reciprocity in the Two port network with ABCD parameter representation.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports. In this case it is easy to use the Z parameter definitions of Z_{11} and Z_{22} for the input and output ports respectively and get their values in terms of the ABCD parameters as shown below.

$$\begin{aligned} V_1 &= A.V_2 - B.I_2 \\ &= C.V_2 - D.I_2 \end{aligned}$$

$$Z_{11} = V_1/I_1 \mid I_2=0$$

Applying this in both the equations we get

$$\begin{aligned} Z_{11} &= V_1/I_1 \mid I_2=0 = (A.V_2 - B.I_2)/(C.V_2 - D.I_2) \mid I_2=0 \\ &= (A.V_2 - B.0)/(C.V_2 - D.0) \\ &= (A.V_2)/(C.V_2) = A/C \end{aligned}$$

$$Z_{11} = A/C$$

$$\text{Similarly } Z_{22} = V_2/I_2 \mid I_1=0$$

and using this in the second basic equation $I_1 = C.V_2 - D.I_2$

$$\begin{aligned} \text{we get } 0 &= C.V_2 - D.I_2 \text{ or } C.V_2 = D.I_2 \quad V_2 / \\ I_2 &= D/C \end{aligned}$$

$$Z_{22} = D/C$$

And the condition for symmetry becomes $Z_{11} = Z_{22}$ i.e. $A/C = D/C$ or $A = D$

Hence $A = D$ is the condition for Symmetry in ABCD parameter representation.

h parameter representation:

Condition for reciprocity :

Let us take a two port network with h parameter defining equations as given below:

$$\begin{aligned} V_1 &= h_{11} \cdot I_1 + h_{12} \cdot V_2 \\ &= h_{21} \cdot I_1 + h_{22} \cdot V_2 \end{aligned}$$

First we will get an expression for the ratio of response (I_2) to the excitation (V_1) in terms of the **h parameters** by giving excitation (V_1) at the input port and considering the current (I_2) in the output port as response (by short circuiting the output port i.e. setting V_2 as zero)

Then the first equation in the **h** parameter defining equations would become

$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot 0 = h_{11} \cdot I_1$$

And in the same condition the second equation in the **h** parameter defining equations would become

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot 0 = h_{21} \cdot I_1$$

Dividing the second equation by the first equation we get

$$I_2/V_1 = (h_{21} \cdot I_1) / (h_{11} \cdot I_1) = h_{21}/h_{11}$$

Now the excitation and the response ports are interchanged and then we will get an expression for the ratio of response (I_1) to the excitation (V_2) in terms of the **h parameters** by giving excitation (V_2) at the output port and considering the current (I_1) in the input port as response (by short circuiting the input port i.e. setting V_1 as zero)

Then the first equation in **h** parameter defining equations would become

$$0 = h_{11} \cdot I_1 + h_{12} \cdot V_2 \quad \text{i.e.} \quad h_{11} \cdot I_1 = -h_{12} \cdot V_2$$

$$\text{i.e.} \quad I_1 / V_2 = -h_{12} / h_{11}$$

Assuming the input excitations **V₁** and **V₂** to be the same, then the condition for the out responses **I₁** and **I₂** to be equal would be

$$I_1 / V_2 = I_2 / V_1$$

$$\text{i.e.} \quad -h_{12} / h_{11} = h_{21} / h_{11}$$

$$\text{i.e.} \quad h_{12} = -h_{21}$$

And hence [**h₁₂ = - h₂₁**] is the condition for the reciprocity in the Two port network with **h** parameter representation.

Condition for symmetry:

To get this condition we need to get expressions related to the input and output ports. In this case also it is easy to use the **Z** parameter definitions of **Z₁₁** and **Z₂₂** for the input and output ports respectively and get their values in terms of the **h** parameters as shown below.

h parameter equations are: $V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$$

First let us get **Z₁₁** :

$$Z_{11} = V_1 / I_1 \mid I_2 = 0$$

$$= h_{11} + h_{12} \cdot V_2 / I_1$$

Applying the condition $I_2=0$ in the equation 2 we get

$$0 = h_{21} \cdot I_1 + h_{22} \cdot V_2 \text{ i.e } -h_{21} \cdot I_1 = h_{22} \cdot V_2$$

$$\text{or } V_2 = I_1 (-h_{21} / h_{22})$$

Now substituting the value of $V_2 = I_1 (-h_{21} / h_{22})$ in the above first expression for V_1 we get

$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot I_1 (-h_{21} / h_{22})$$

$$\text{Or } V_1 / I_1 = (h_{11} \cdot h_{22} - h_{12} \cdot h_{21}) / h_{22} = \Delta h / h_{22}$$

$$\text{Or } Z_{11} = \Delta h / h_{22}$$

Where $\Delta h = (h_{11} \cdot h_{22} - h_{12} \cdot h_{21})$ Now let us

get Z_{22} :

$$Z_{22} = V_2 / I_2 \mid I_1 = 0$$

Applying the condition $I_1 = 0$ in the second equation we get

$$I_2 = h_{21} \cdot 0 + h_{22} \cdot V_2 \text{ i.e } V_2 / I_2 = 1 / h_{22}$$

$$\text{And } Z_{22} = 1 / h_{22}$$

Hence the condition for symmetry $Z_{11} = Z_{22}$ becomes $(\Delta h / h_{22}) = (1 / h_{22})$ i.e $\Delta h = 1$

Hence $\Delta h = 1$ is the condition for symmetry in h parameter representation.

Table Summary of conditions for reciprocity and symmetry for Two port networks in terms of all four parameters.

Parameter	Condition for reciprocity	Condition for symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{21}$
h	$h_{12} = -h_{21}$	$\Delta h = 1$
$ABCD$	$AD - BC = 1$	$A = D$

Different types of interconnections of two port networks:

Series Connection:

Though here only two networks are considered, the result can be generalized for any number of two port networks connected in series.

Refer the figure below where two numbers of two port networks **A** and **B** are shown connected in series. All the input and output currents & voltages with directions and polarities are shown.

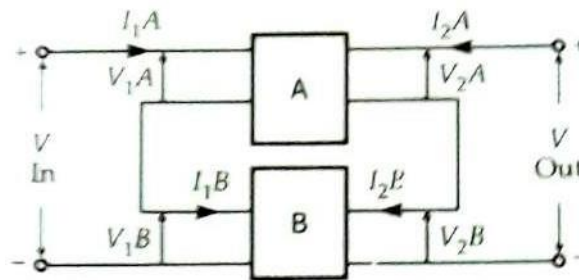


Fig : Series connection of two numbers of Two Port Networks

Open circuit Impedance parameters (Z) are used in characterizing the Series connected Two port Networks. The governing equations with Z parameters are given below:

For network A :

$$V_{1A} = Z_{11A} I_{1A} + Z_{12A} I_{2A}$$

$$V_{2A} = Z_{21A} I_{1A} + Z_{22A} I_{2A}$$

And for network B:

$$V_{1B} = Z_{11B} I_{1B} + Z_{12B} I_{2B}$$

$$V_{2B} = Z_{21B} I_{1B} + Z_{22B} I_{2B}$$

Referring to the figure above the various voltage and current relations are:

$$I_1 \equiv I_{1A} \equiv I_{1B}$$

$$I_2 \equiv I_{2A} \equiv I_{2B}$$

$$V_2 = V_{2A} + V_{2B}$$

$$V_1 = V_{1A} + V_{1B}$$

Now substituting the above basic defining equations for the two networks into the above expressions for V_1 and V_2 and using the above current equalities we get:

$$V_1 = V_{1A} + V_{1B}$$

$$= (Z_{11A} I_{1A} + Z_{12A} I_{2A}) + Z_{11B} I_{1B} + Z_{12B} I_{2B}$$

$$= I_1 (Z_{11A} + Z_{11B}) + I_2 (Z_{12A} + Z_{12B})$$

And similarly

$$V_2 = V_{2A} + V_{2B}$$

$$= (Z_{21A} I_{1A} + Z_{22A} I_{2A}) + (Z_{21B} I_{1B} + Z_{22B} I_{2B})$$

$$V_2 = I_1 (Z_{21A} + Z_{21B}) + I_2 (Z_{22A} + Z_{22B})$$

Thus we get for two numbers of series connected two port networks:

$$V_1 = (Z_{11A} + Z_{11B}) I_1 + (Z_{12A} + Z_{12B}) I_2$$

$$V_2 = (Z_{21A} + Z_{21B}) I_1 + (Z_{22A} + Z_{22B}) I_2$$

Or in matrix form:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11A} + Z_{11B} & Z_{12A} + Z_{12B} \\ Z_{21A} + Z_{21B} & Z_{22A} + Z_{22B} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus it can be seen that the Z parameters for the series connected two port networks are the sum of the Z parameters of the individual two port networks.

Cascade connection:

In this case also though here only two networks are considered, the result can be generalized for any number of two port networks connected in cascade.

Refer the figure below where two numbers of two port networks **X** and **Y** are shown connected in cascade. All the input and output currents & voltages with directions and polarities are shown.

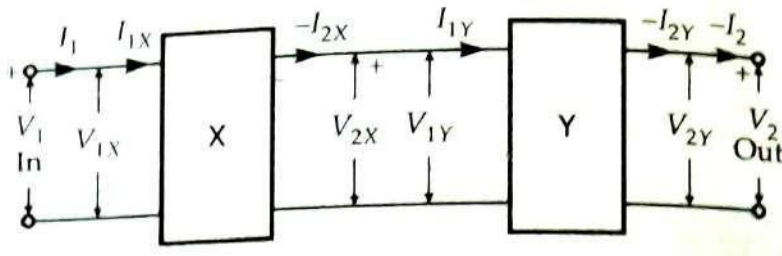


Fig 5.8: Two numbers of two port networks connected in cascade

Transmission (ABCD) parameters are easily used in characterizing the cascade connected Two port Networks. The governing equations with transmission parameters are given below:

For network X:

$$\begin{aligned} V_{1X} &= A_X V_{2X} - B_X I_{2X} \\ I_{1X} &= C_X V_{2X} - D_X I_{2X} \end{aligned}$$

And for network Y:

Referring to the figure above the various voltage and current relations are:

$$\begin{aligned} I_1 &= I_{1X} ; -I_{2X} = I_{1Y} ; I_2 = I_{2Y} \\ V_1 &= V_{1X} ; V_{2X} = V_{1Y} ; V_2 = V_{2Y} \end{aligned}$$

Then the overall transmission parameters for the cascaded network in matrix form will become

$$\begin{aligned}\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} &= \begin{bmatrix} V_{1X} \\ I_{1X} \end{bmatrix} = \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} V_{2X} \\ -I_{2X} \end{bmatrix} \\ &= \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} V_{1Y} \\ I_{1Y} \end{bmatrix} \\ &= \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ C_Y & D_Y \end{bmatrix} \begin{bmatrix} V_{2Y} \\ -I_{2Y} \end{bmatrix} \\ &= \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ C_Y & D_Y \end{bmatrix} \begin{bmatrix} V_Y \\ -I_Y \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_Y \\ -I_Y \end{bmatrix}$$

Where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix} \begin{bmatrix} A_Y & B_Y \\ D_Y & D_Y \end{bmatrix}$$

Thus it can be seen that the overall ABCD Parameter matrix of cascaded two Port Networks is the product of the ABCD matrices of the individual networks.

Parallel Connection:

Though here only two networks are considered, the result can be generalized for any number of two port networks connected in parallel.

Refer the figure below where two numbers of two port networks **A** and **B** are shown connected in parallel. All the input and output currents & voltages with directions and polarities are shown.

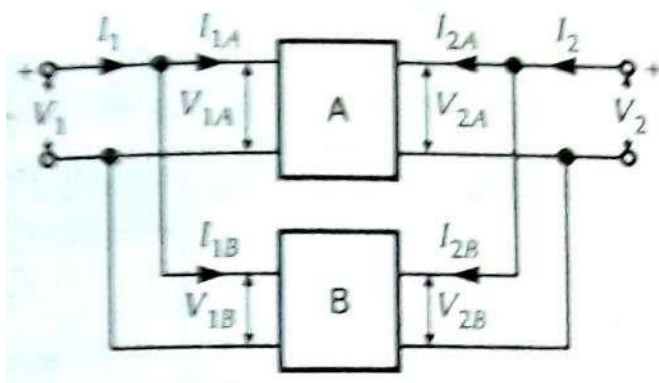


Fig 5.9: Parallel connection of two numbers of Two Port Networks

Short circuit admittance (Y) parameters are easily used in characterizing the parallel connected Two port Networks. The governing equations with Y parameters are given below:

For network A:

$$\begin{aligned} I_{1A} &= Y_{11A} V_{1A} + Y_{12A} V_{2A} \\ I_{2A} &= Y_{21A} V_{1A} + Y_{22A} V_{2A} \end{aligned}$$

And for network B:

$$\begin{aligned} I_{1B} &= Y_{11B} V_{1B} + Y_{12B} V_{2B} \\ I_{2B} &= Y_{21B} V_{1B} + Y_{22B} V_{2B} \end{aligned}$$

Referring to the figure above the various voltage and current relations are:

$$\begin{aligned} V_1 &= V_{1A} = V_{1B}; V_2 = V_{2A} = V_{2B} \\ I_1 &= I_{1A} + I_{1B}; I_2 = I_{2A} + I_{2B} \end{aligned}$$

Thus

$$\begin{aligned} I_1 &= I_{1A} + I_{1B} \\ &= (Y_{11A} V_{1A} + Y_{12A} V_{2A}) + (Y_{11B} V_{1B} + Y_{12B} V_{2B}) \\ &= (Y_{11A} + Y_{11B}) V_1 + (Y_{12A} + Y_{12B}) V_2 \\ I_2 &= I_{2A} + I_{2B} \\ &= (Y_{21A} V_{1A} + Y_{22A} V_{2A}) + (Y_{21B} V_{1B} + Y_{22B} V_{2B}) \\ &= (Y_{21A} + Y_{21B}) V_1 + (Y_{22A} + Y_{22B}) V_2 \end{aligned}$$

Thus we finally obtain the Y parameter equations for the combined network as:

$$\begin{aligned} I_1 &= (Y_{11A} + Y_{11B}) V_1 + (Y_{12A} + Y_{12B}) V_2 \\ I_2 &= (Y_{21A} + Y_{21B}) V_1 + (Y_{22A} + Y_{22B}) V_2 \end{aligned}$$

And in matrix notation it will be:

$$\begin{bmatrix} I_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} Y_{11A} + Y_{11B} & Y_{12A} + Y_{12B} \\ Y_{21A} + Y_{21B} & Y_{22A} + Y_{22B} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus it can be seen that the overall Y parameters for the parallel connected two port networks are the sum of the Y parameters of the individual two port networks.

Image impedances in terms of ABCD parameters:

Image impedances Z_{i1} and Z_{i2} of a two port network as shown in the figure below are defined as two values of impedances such that :

- When port two is terminated with an impedance Z_{i2} , the input impedance as seen from Port one is Z_{i1} and
- When port one is terminated with an impedance Z_{i1} , the input impedance as seen from Port two is Z_{i2}

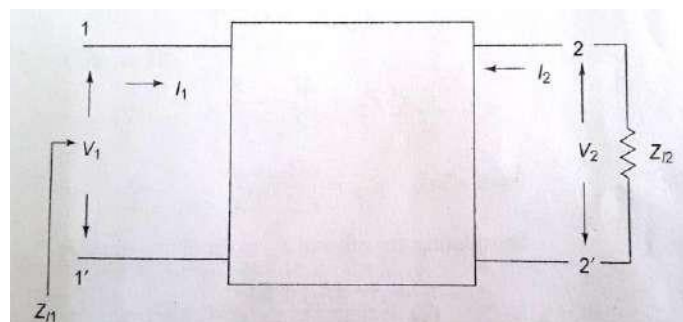


Figure 5.10: pertaining to condition (a) above

Corresponding Relations are : $Z_{i1} = V_1 / I_1$ and $Z_{i2} = V_2 / -I_2$

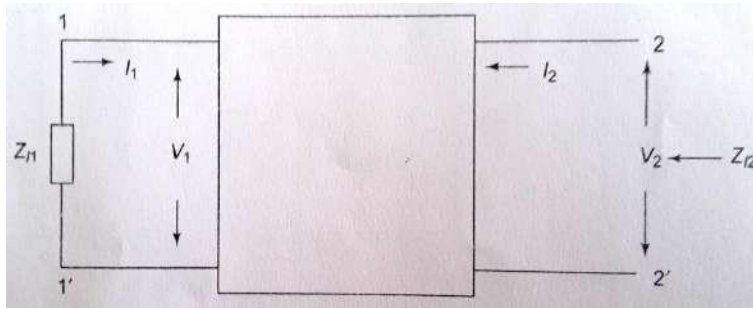


Figure 5.10: pertaining to condition (b) above

Corresponding Relations are : $Z_{i1} = V_1 / -I_1$ and $Z_{i2} = V_2 / I_2$

Such Image impedances in terms of ABCD parameters for a two port network are obtained below:

The basic defining equations for a two port network with ABCD parameters are :

$$\begin{aligned} V_1 &= A.V_2 - B.I_2 \\ I_1 &= C.V_2 - D.I_2 \end{aligned}$$

First let us consider condition (a).

Dividing the first equation with the second equation we get

$$Z_{i1} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

But we also have $Z_{i2} = V_2 / -I_2$ and so $V_2 = -Z_{i2}I_2$. Substituting this value of V_2 in the above we get

$$Z_{i1} = \frac{-AZ_{i2} - B}{-CZ_{i2} - D} = \frac{AZ_{i2} + B}{CZ_{i2} + D}$$

The basic governing equations $[V_1 = A.V_2 - B.I_2]$ and $[I_1 = C.V_2 - D.I_2]$ are manipulated to get

$$V_2 = \frac{DV_1}{AD - BC} - \frac{BI_1}{AD - BC}$$

$$I_2 = \frac{CV_1}{AD - BC} - \frac{AI_1}{AD - BC}$$

$$Z_{i2} = \frac{V_2}{I_2} = \frac{DV_1 - BI_1}{CV_1 - AI_1}$$

But we also have $Z_{i1} = V_1 / -I_1$ and so $V_1 = -Z_{i1}I_1$. Substituting this value of V_1 in the above we get:

$$Z_{i2} = \frac{DZ_{i1} + B}{CZ_{i1} + A}$$

Solving the above equations for Z_{i1} and Z_{i2} we get :

$$Z_{i1} = \sqrt{\frac{AB}{CD}} ; \quad Z_{i2} = \sqrt{\frac{BD}{AC}}$$

Important formulae, Equations and Relations:

- **Basic Governing equations in terms of the various Parameters:**

- **Z Parameters :** $V_1 = Z_{11}I_1 + Z_{12}I_2$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

- **Y Parameters:** $I_1 = Y_{11}V_1 + Y_{12}V_2$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

- **h Parameters:** $V_1 = h_{11}.I_1 + h_{12}.V_2$

$$I_2 = h_{21}.I_1 + h_{22}.V_2$$

ABCD Parameters: $V_1 = A.V_2 - B.I_2$

$$I_1 = C.V_2 - D.I_2$$

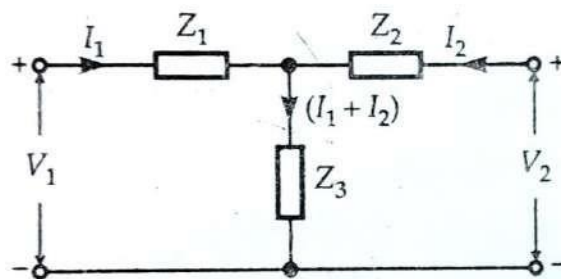
- **Conditions for Reciprocity and symmetry for Two Port Networks in terms of the various parameters :**

Parameter	Condition for reciprocity	Condition for symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{21}$
h	$h_{12} = -h_{21}$	$\Delta h = 1$
$ABCD$	$AD - BC = 1$	$A = D$

- **Relations of Interconnected two port Networks :**
- The overall **Z parameters** for the **series** connected two port networks are the **sum of the Z parameters of the individual two port networks.**
- The overall **Y parameters** for the **parallel** connected two port networks are the **sum of the Y parameters of the individual two port networks.**
- The overall **ABCD Parameter matrix** of **cascaded** two Port Networks is the **product of the ABCD matrices of the individual networks.**

Illustrative problems :

Example 1: Find the Z Parameters of the following Two Port Network and draw it's equivalent circuit in terms of Z_1 Z_2 and Z_3 .



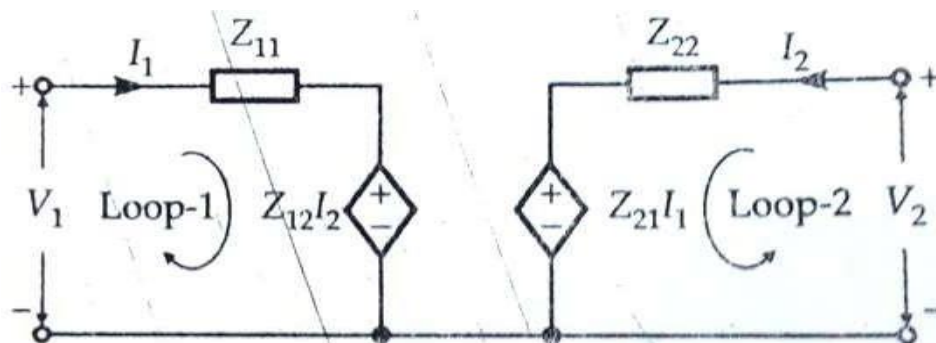
Solution: Applying KVL to the above circuit in the two loops, with the current notation as shown, the loop equations for V_1 and V_2 can be written as :

$$\begin{aligned}
 V_1 &= I_1 Z_1 + (I_1 + I_2) Z_3 \\
 \text{or } V_1 &= (Z_1 + Z_3) I_1 + Z_3 I_2 \quad \dots(i) \\
 \text{and } V_2 &= I_2 Z_2 + (I_2 + I_1) Z_3 \\
 \text{or } V_2 &= Z_3 I_1 + (Z_2 + Z_3) I_2 \quad \dots(ii)
 \end{aligned}$$

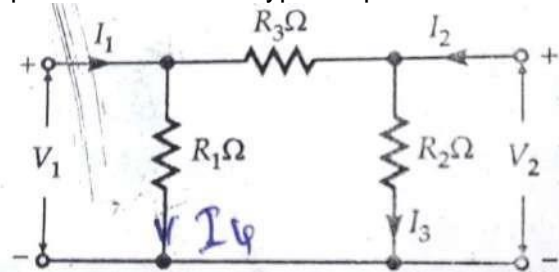
Comparing the equations (i) and (ii) above with the standard expressions for the Z parameter equations we get:

$$\begin{aligned}
 Z_{11} &= Z_1 + Z_3; \quad Z_{12} = Z_3; \\
 Z_{21} &= Z_3; \quad Z_{22} = Z_2 + Z_3
 \end{aligned}$$

Equivalent circuit in terms of Z_1 , Z_2 and Z_3 is shown below.



Example 2: Determine the Z parameters of the π type two port network shown in the figure below.



Solution:

From the basic Z parameter equations We know that

$$Z_{11} = V_1/I_1 \mid I_2=0 \quad Z_{12}$$

$$= V_1/I_2 \mid I_1=0 \quad Z_{21} =$$

$$V_2/I_1 \mid I_2=0 \quad Z_{22} =$$

$$V_2/I_2 \mid I_1=0$$

We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$

1. We can observe that $Z_{11} = V_1/I_1$ with $I_2=0$ is the parallel combination of R_1 and $(R_2 + R_3)$.

$$\therefore Z_{11} = R_1 (R_2 + R_3) / (R_1 + R_2 + R_3)$$

2. $Z_{21} = V_2/I_1 \mid I_2=0$

By observing the network we find that the current I_1 is dividing into I_3 and I_4 as shown in the figure where I_3 is flowing through R_2 (and R_3 also since $I_2=0$)

Hence $V_2 = I_3 \times R_2$

From the principle of current division we find that $I_3 = I_1 \cdot R_1 / (R_1 + R_2 + R_3)$ Hence

$$V_2 = I_3 \times R_2 = [I_1 \cdot R_1 / (R_1 + R_2 + R_3)] \cdot R_2 = I_1 \cdot R_1 R_2 / (R_1 + R_2 + R_3)$$

$$\text{And } V_2/I_1 = R_1 R_2 / (R_1 + R_2 + R_3)$$

$$\therefore Z_{21} = R_1 R_2 / (R_1 + R_2 + R_3)$$

Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$. $Z_{12} =$

$$V_1/I_2 \mid I_1=0$$

By observing the network we find that the current I_2 is now dividing into I_3 and I_4 as shown in the figure where I_4 is flowing through R_1 (and R_3 also since $I_1 = 0$)

Hence $V_1 = I_4 \times R_1$

Again from the principle of current division we find that $I_4 = I_2 \cdot R_2 / (R_1 + R_2 + R_3)$ Hence V_1

$$= I_4 \times R_1 = [I_2 \cdot R_2 / (R_1 + R_2 + R_3)] \cdot R_1 = I_2 \cdot R_1 R_2 / (R_1 + R_2 + R_3)$$

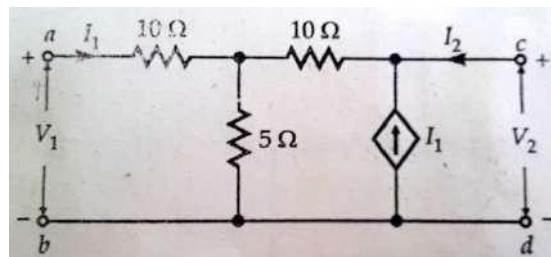
$$\text{And } V_1 / I_2 = R_1 R_2 / (R_1 + R_2 + R_3)$$

$$\therefore Z_{12} = R_1 R_2 / (R_1 + R_2 + R_3)$$

4. We can again observe that $Z_{22} = V_2 / I_2$ with $I_1 = 0$ is the parallel combination of R_2 and $(R_1 + R_3)$

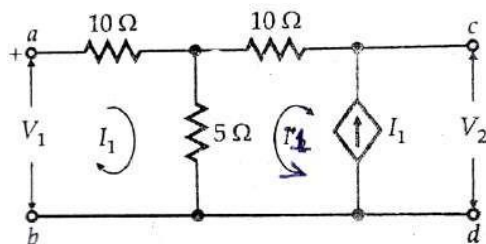
$$\therefore Z_{22} = R_2 (R_1 + R_3) / (R_1 + R_2 + R_3)$$

Example 3 : Determine the Z parameters of the network shown in the figure below.



1). We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

With this condition the circuit is redrawn as shown below.



Since the current source is there in the second loop which is equal to I_1 and I_2 is zero, only current I_1 flows through the right hand side resistance of 10Ω and both currents I_1 (both loop currents) pass through the resistance of 5Ω as shown in the redrawn figure .

Now the equation for loop one is given by :

$$V_1 = 10 \times I_1 + 5 (2 I_1) = 20 I_1 \text{ and } V_1/I_1 = 20\Omega$$

$$\therefore V_1/I_1 \mid I_2=0 = Z_{11} = 20\Omega$$

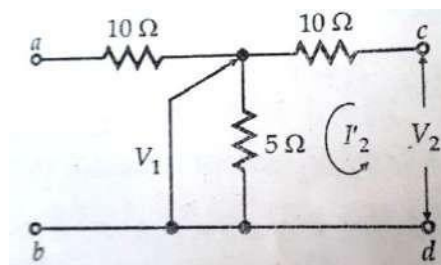
Next the equation for loop two is given by :

$$V_2 = 10 \times I_1 + 5 (2 I_1) = 20 I_1 \text{ and } V_2/I_1 = 20\Omega$$

$$\therefore V_2/I_1 \mid I_2=0 = Z_{21} = 20\Omega$$

2). Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$ (input open circuited)

With this condition the circuit is redrawn as shown below.



Now since the current I_1 is zero ,the current source of I_1 would no longer be there in the output loop and it is removed as shown in the redrawn figure. Further since input current $I_1=0$,there would be no current in the input side 10Ω and the same current I_2 only flows through common resistance of 5Ω and output side resistance of 10Ω .With these conditions incorporated, now we shall rewrite the two loop equations (for input V_1 and output V_2) to get **Z_{12} and Z_{22}**

Equation for loop one is given by :

$$V_1 = 5 I_2 \text{ and } V_1/I_2 = 5\Omega$$

$$\therefore V_1/I_2 \mid I_1=0 = Z_{12} = 5\Omega$$

And the equation for loop two is given by:

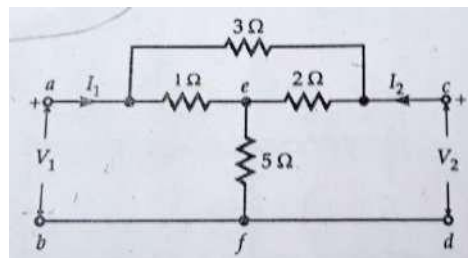
$$V_2 = 10 \times I_2 + 5 \times I_2 = 15 I_2$$

$$\text{and } V_2/I_2 = 15 \Omega$$

$$\therefore V_2/I_2 \mid I_1=0 = Z_{22} = 15 \Omega$$

$$\text{Finally: } Z_{11} = 20 \Omega ; \quad Z_{12} = 5 \Omega ; \quad Z_{21} = 20 \Omega ; \quad Z_{22} = 15 \Omega$$

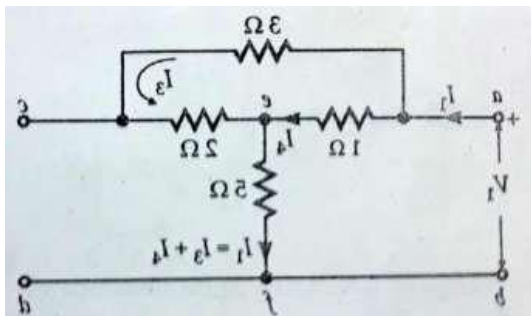
Example 4: Obtain the open circuit parameters of the Bridged T network shown in the figure below.



Open circuit parameters are same as Z parameters.

1). We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

With this condition the circuit is redrawn as shown below.



From the inspection of the figure in this condition it can be seen that (since I_2 is zero) the two resistances i.e the bridged arm of 3Ω and output side resistance of 2Ω are in series and together are in parallel with the input side resistance of 1Ω .

Hence the loop equation for V_1 can be written as:

$$V_1 = I_1 \times [(3+2) \parallel 1 + 5] = I_1 \times 35/6 \text{ and } V_1/I_1 = 35/6$$

$$\therefore V_1/I_1 \mid I_2=0 = Z_{11} = 35/6\Omega$$

Next the loop equation for V_2 can be written as :

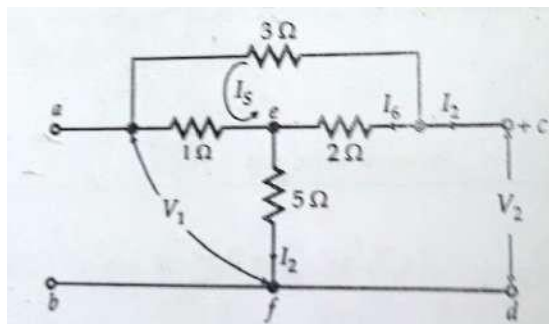
$$V_2 = I_3 \times 2 + I_1 \times 5$$

But we know from the principle of current division that the current $I_3 = I_1 \times [1/(1+2+3)] = I_1 \times 1/6$ Hence $V_2 = I_1 \times 1/6 \times 2 + I_1 \times 5 = I_1 \times 16/3$ and $V_2/I_1 = 16/3 \Omega$

$$\therefore V_2/I_1 \mid I_2=0 = Z_{21} = 16/3 \Omega$$

2). Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$ (input open circuited)

With this condition the circuit is redrawn as shown below.



From the inspection of the figure in this condition it can be seen that (since I_1 is zero) the two resistances i.e. the bridged arm of 3Ω and input side resistance of 1Ω are in series and together are in parallel with the output side resistance of 2Ω . Further $I_2 = I_5 + I_6$

Hence the loop equation for V_1 can be written as: $V_1 =$

$$I_5 \times 1 + I_2 \times 5$$

But we know from the principle of current division that the current $I_5 = I_2 \times [2/(1+2+3)] = I_2 \times 1/3$ Hence $V_1 = I_2 \times$

$$1/3 \times 1 + I_2 \times 5 = I_2 \times 16/3 \text{ and } V_1 / I_2 = 16/3 \Omega$$

$$\therefore V_1 / I_2 \mid I_1=0 = Z_{12} = 16/3 \Omega$$

Next the loop equation for V_2 can be written as:

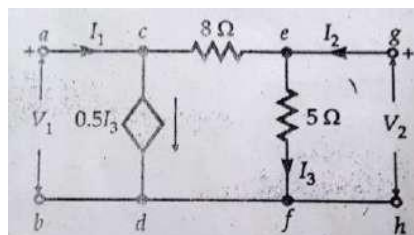
$$V_2 = I_6 \times 2 + I_2 \times 5$$

But we know from the principle of current division that the current $I_6 = I_2 \times [1/(1+2+3)] = I_2 \times (3+1)/6 = (I_2 \times 2/3)$

Hence $V_2 = I_2 \times (2/3) \times 2 + I_2 \times 5 = I_2 \times 19/3$ and $V_2 / I_2 = 19/3$

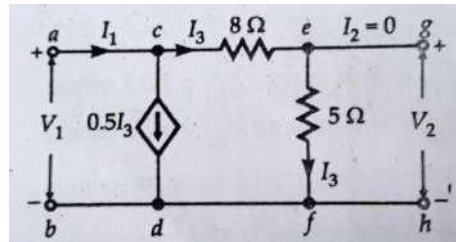
$$\therefore V_2 / I_2 \mid I_2=0 = Z_{22} = 19/3 \Omega$$

Example 5 : Obtain Z parameters of the following π network with a controlled current source of $0.5 I_3$ in the input port.



1). We will first find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

With this condition the circuit is redrawn as shown below.



In this condition we shall first apply Kirchhoff's current law to the node 'c':

Then $I_1 = 0.5I_3 + I_3$ (I_3 being the current through the resistances of 8Ω and 5Ω)

i.e. $I_1 = 0.5I_3 + I_3$ or $I_1 = 1.5I_3$ or $I_3 = I_1/1.5$ i.e. $I_3 = (2/3)I_1$

Now we also observe that $V_1 = I_3(8+5) = 13 \cdot I_3$

Using the value of $I_3 = (2/3)I_1$ into the above expression we get $V_1 =$

$13(2/3)I_1$ and $V_1/I_1 = 26/3 = 8.67$

$$\therefore V_1/I_1 \mid I_2=0 = Z_{11} = 8.67\Omega$$

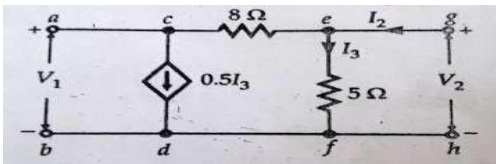
Next we also observe that $V_2 = 5 \cdot I_3$ and substituting the above value of $I_3 = (2/3)I_1$ into this expression for V_2 we get:

$V_2 = 5 \cdot I_3$ i.e. $V_2 = 5 \cdot (2/3)I_1$ i.e. $V_2/I_1 = 10/3 = 3.33\Omega$

$$\therefore V_2/I_1 \mid I_2=0 = Z_{21} = 3.33\Omega$$

2). Next we will find out Z_{12} and Z_{22} which are given by the common condition $I_1 = 0$ (input open circuited)

With this condition the circuit is redrawn as shown below.



In this condition now we shall first apply Kirchhoff's current law to the node 'e':

Then $I_2 = 0.5I_3 + I_3$ ($0.5I_3$ being the current through the resistance of 8Ω and I_3 being the current through the resistances of 5Ω)

$$\text{i.e. } I_2 = 0.5I_3 + I_3 \text{ or } I_2 = 1.5I_3 \text{ or } I_3 = I_2/1.5 \text{ i.e. } I_3 = (2/3)I_2$$

Now we also observe that $V_1 = (-0.5I_3 \times 8 + I_3 \times 5) = I_3$ (it is to be noted here carefully that -sign is to be taken before $0.5I_3 \times 8$ since the current flows through the resistance of 8Ω now in the reverse direction).

Using the value of $I_3 = (2/3)I_2$ into the above expression for V_1 we get $V_1 =$

$$(2/3)I_2 \quad \text{and } V_1/I_2 = 0.67$$

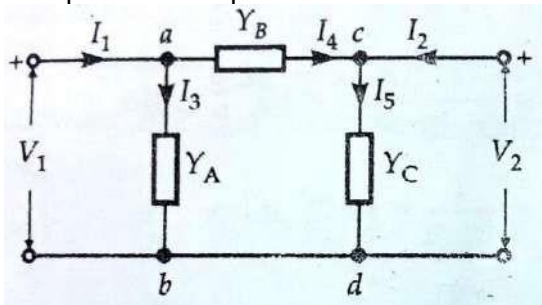
$$\therefore V_1/I_2 \mid I_1=0 = Z_{12} = 0.67\Omega$$

Next we also observe that $V_2 = 5 \cdot I_3$ and substituting the above value of $I_3 = (2/3)I_2$ into this expression for V_2 we get :

$$V_2 = 5 \cdot I_3 \text{ i.e. } V_2 = 5 \cdot (2/3)I_2 \text{ i.e. } V_2 / I_2 = 10/3 = 3.33\Omega$$

$$\therefore V_2/I_2 \mid I_1=0 = Z_{21} = 3.33\Omega$$

Example 6 : Find the Y parameters of the following π type two port network and draw its Y parameter equivalent circuit in terms of the given circuit parameters.



Applying KCL at node (a) we get

$$I_1 = I_3 + I_4$$

$$I_1 = V_1 Y_A + (V_1 - V_2) Y_B$$

$$I_1 = V_1 (Y_A + Y_B) + (-Y_B) V_2 \quad \dots (i)$$

Similarly applying KCL to node (c) we get

$$I_2 = I_5 - I_4$$

$$I_2 = V_2 Y_C - (V_1 - V_2) Y_B$$

$$I_2 = (-Y_B) V_1 + (Y_C + Y_B) V_2 \quad \dots (ii)$$

Comparing the equations (i) and (ii) above with the standard expressions for the Y parameter equations we get:

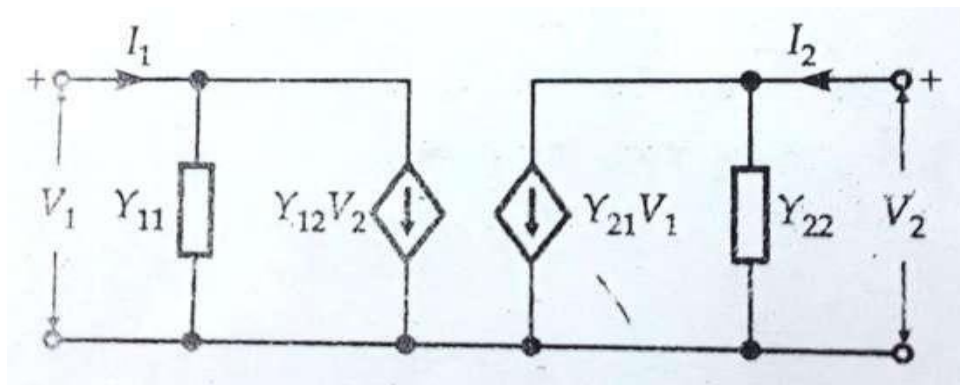
$$Y_{11} = (Y_A + Y_B); Y_{12} = -Y_B$$

$$Y_{21} = -Y_B; Y_{22} = Y_C + Y_B.$$

Observing the equations (i) and (ii) above we find that :

- The terms $V_1(Y_A + Y_B)$ and $V_2(Y_C + Y_B)$ are the currents through the admittances Y_{11} and Y_{22} and
- The terms $-Y_B \cdot V_2$ and $-Y_B \cdot V_1$ are the dependent current sources in the input and the output ports respectively.

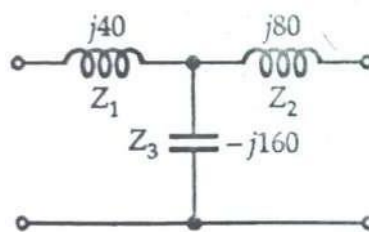
These observations are reflected in the equivalent circuit shown below.



In the above figure $Y_{11} = (Y_A + Y_B)$ & $Y_{22} = (Y_C + Y_B)$ are the admittances and

$Y_{12} \cdot V_2 = -Y_B \cdot V_2$ & $Y_{21} \cdot V_1 = -Y_B \cdot V_1$ are the dependent current sources

Example 7: Find the Y parameters of the following network



Solution: We will solve this problem in two steps.

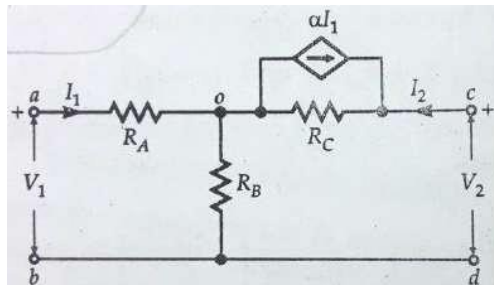
1. We shall first express the Z parameters of the given T network in terms of the impedances Z_1 , Z_2 and Z_3 using the standard formulas we already know and substitute the given values of Z_1 , Z_2 and Z_3 .

$$\begin{aligned} Z_{11} &= Z_1 + Z_3 = -j120 ; \\ Z_{12} &= Z_3 = -j160 \\ Z_{21} &= Z_3 = -j160 ; \\ Z_{22} &= Z_2 + Z_3 = -j80 \end{aligned}$$

2. Then convert the values of the Z parameters into Y parameters i.e express the Y parameters in terms of Z parameters using again the standard relationships.

$$\begin{aligned}
 Y_{11} &= \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}} \\
 &= \frac{-j 80}{(-j 120)(-j 80) - (-j 160)^2} \\
 &= \frac{-j 80}{16,000} = \frac{-j}{200} \text{ mho.} \\
 Y_{12} = Y_{21} &= \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}Z_{21}} \\
 &= \frac{j 160}{16,000} = \frac{j}{100} \text{ mho.} \\
 Y_{22} &= \frac{Z_{11}}{Z_{11}Z_{22} - Z_{12}Z_{21}} \\
 &= \frac{-j 120}{16,000} = \frac{-j}{133.33} \text{ mho.}
 \end{aligned}$$

Example 8: Find the 'h' parameters of the network shown below. (fig12.34)



First let us write down the basic 'h' parameter equations and give the definitions of the 'h' parameters.

$$\begin{aligned} V_1 &= h_{11} \cdot I_1 + h_{12} \cdot V_2 \\ I_2 &= h_{21} \cdot I_1 + h_{22} \cdot V_2 \end{aligned}$$

$$h_{11} = V_1/I_1 \text{ with } V_2=0$$

$$h_{21} = I_2/I_1 \text{ with } V_2=0$$

$$h_{12} = V_1/V_2 \text{ with } I_1=0$$

$$h_{22} = I_2/V_2 \text{ with } I_1=0$$

Now

1). We will first find out h_{11} and h_{21} which are given by the common condition $V_2=0$ (Output short circuited)

In this condition it can be observed that the resistance R_C and the current source αI_1 become parallel with resistance R_B .

For convenience let us introduce a temporary variable V as the voltage at the node 'o'. Then the current through the parallel combination of R_B and R_C would be equal to

$$\frac{V}{\frac{R_B R_C}{R_B + R_C}} = \frac{V(R_B + R_C)}{R_B R_C}$$

Then applying KCL at the node 'o' we get

$$\begin{aligned} I_1 &= \frac{V(R_B + R_C)}{R_B R_C} + \alpha I_1 \\ I_1(1 - \alpha) &= \frac{V(R_B + R_C)}{R_B R_C} \\ \therefore V &= \frac{(1 - \alpha) I_1 R_B R_C}{(R_B + R_C)} \end{aligned}$$

Next applying KVL at input port we get $V_1 = I_1 R_A + V$ and $V_1/I_1 = R_A + V/I_1$. Now using the value of V we obtained above in this expression for V_1/I_1 we get

$$h_{11} = \frac{V_1}{I_1} = R_A + \frac{(1-\alpha) R_B R_C}{R_B + R_C}$$

$$= \frac{R_A (R_B + R_C) + (1-\alpha) R_B R_C}{R_B + R_C} \text{ ohm.}$$

Again from inspection of the figure above it is evident that

$$I_2 = - \left(\alpha I_1 + \frac{V}{R_C} \right)$$

$$I_2 = -\alpha I_1 - \frac{(1-\alpha) I_1 R_B}{(R_B + R_C)}$$

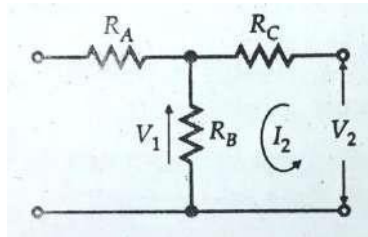
Therefore

$$h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2=0} = -\alpha - \frac{(1-\alpha) R_B}{(R_B + R_C)}$$

$$= -\frac{(\alpha R_C + R_B)}{(R_B + R_C)}.$$

2). Next we will find out h_{12} and h_{22} which are given by the common condition $I_1 = 0$ (Input open circuited)

Now since I_1 is zero, the current source disappears and the circuit becomes simpler as shown in the figure below.



Now applying KVL at the output port we get:

$$V_2 = I_2 (R_B + R_C)$$

$$\left. \frac{I_2}{V_2} \right|_{I_1=0} = h_{22} = \left(\frac{1}{R_B + R_C} \right) \text{ mho.}$$

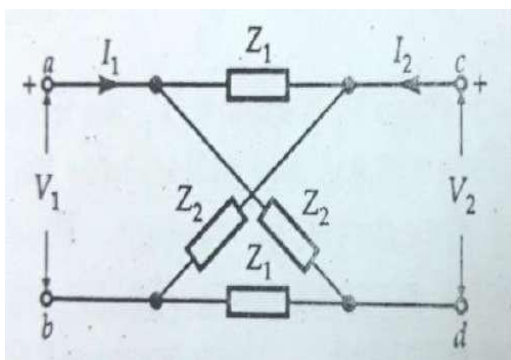
Again under this condition:

$$V_1 = I_2 R_B$$

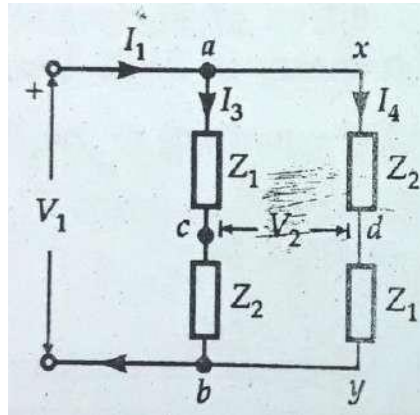
$$\therefore h_{12} = \left. \frac{V_1}{V_2} \right|_{I_1=0} = \frac{I_2 R_B}{I_2 (R_B + R_C)}$$

$$= \frac{R_B}{R_B + R_C}.$$

Example 9 : Z parameters of the lattice network shown in the figure below.



First we shall redraw the given lattice network in a simpler form for easy analysis as shown below.



We will then find out Z_{11} and Z_{21} which are given by the common condition $I_2 = 0$ (Output open circuited)

It can be observed that the impedances in the two arms 'ab' and 'xy' are same i.e. $Z_1 + Z_2$ and their parallel combination is $(Z_1 + Z_2)/2$

Hence applying KVL at the input port we get

$$V_1 = I_1 \left(\frac{Z_1 + Z_2}{2} \right)$$

$$\frac{V_1}{I_1} = Z_{11} \big|_{I_2=0} = \frac{Z_1 + Z_2}{2}$$

Next we find that

$$\begin{aligned} V_2 &= V_c - V_d = (V_1 - I_3 Z_1) - (V_1 - I_4 Z_2) \\ &= I_4 Z_2 - I_3 Z_1 \end{aligned}$$

(V_c and V_d being the potentials at points 'c' and 'd')

It can also be observed from the simplified circuit that the currents I_3 and I_4 through the branches 'ab' and 'xy' are equal since the branch impedances are same and same voltage V_1 is applied across both the branches. Hence the current I divides equally as I_3 and I_4

$$\text{i.e } I_3 = I_4 = I/2$$

Now substituting these values of I_3 and I_4 in the expression for V_2 above:

$$V_2 = \frac{I_1}{2} \times Z_2 - \frac{I_1}{2} \times Z_1 = \frac{Z_2 - Z_1}{2} \cdot I_1$$

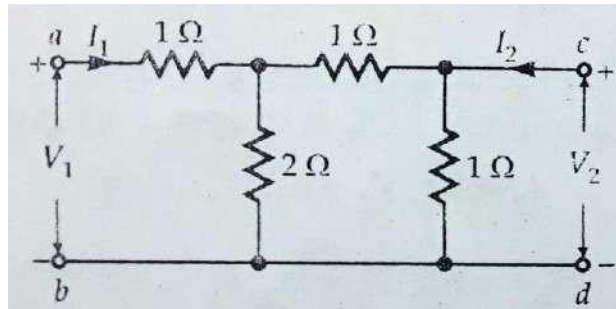
$$\frac{V_2}{I_1} = Z_{21} \Big|_{I_2=0} = \frac{Z_2 - Z_1}{2}$$

As can be seen the circuit is both symmetrical and Reciprocal and hence :

$$Z_{11} = Z_{22} = \frac{Z_1 + Z_2}{2}$$

$$Z_{12} = Z_{21} = \frac{Z_2 - Z_1}{2}$$

Example 10: Find the transmission parameters of the following network (fig 12.51)



First let us write down the basic ABCD parameter equations and give their definitions.

$$\begin{aligned} V_1 &= A.V_2 - B.I_2 \\ I_1 &= C.V_2 - D.I_2 \end{aligned}$$

$$A = V_1/V_2 \text{ with } I_2 = 0$$

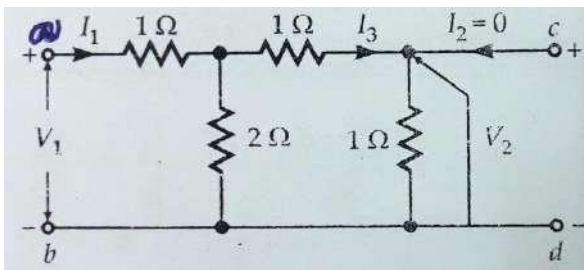
$$C = I_1/V_2 \text{ with } I_2 = 0$$

$$B = V_1/-I_2 \text{ with } V_2 = 0$$

$$D = I_1/-I_2 \text{ with } V_2 = 0$$

1). We will then find out A and C which are given by the common condition $I_2 = 0$ (Output open circuited)

The resulting circuit in this condition is redrawn below.

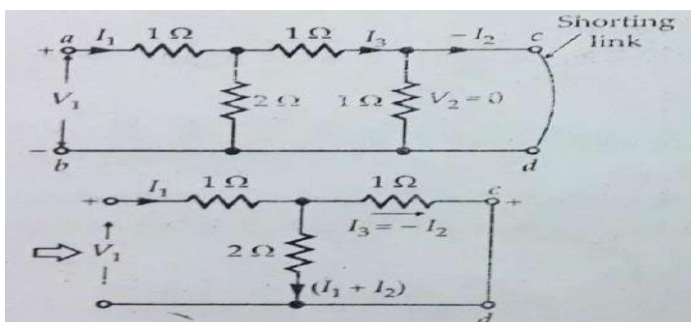


Applying KVL we can write down the two mesh equations and get the values of A and C :

$$\begin{aligned}
 V_1 &= I_1 \times 1 + (I_1 - I_3)2 \\
 \text{or} \quad V_1 &= 3I_1 - 2I_3 \quad \dots(i) \\
 \text{and} \quad 0 &= (I_3 - I_1)2 + I_3(1+1) = 4I_3 - 2I_1 \\
 \therefore I_3 &= \frac{1}{2} I_1 \quad \dots(ii) \\
 \text{Utilising (ii) in (i),} \\
 V_1 &= 3I_1 - 2 \times \frac{1}{2} I_1 = 2I_1 \quad \dots(iii) \\
 \therefore \text{Again,} \\
 V_2 &= I_3 \times 1 = \frac{1}{2} I_1 \quad \dots(iv) \\
 \therefore \left. \frac{I_1}{V_2} \right|_{I_2=0} &= 2 \text{ mho} = C. \\
 \text{Dividing equation (iii) by (iv),} \\
 \left. \frac{V_1}{V_2} \right|_{I_2=0} &= 4 = A
 \end{aligned}$$

2.) Next we will find out B and D which are given by the common condition $V_2 = 0$ (Output short circuited)

The resulting simplified network in this condition is redrawn below.



The voltage at the input port is given by: $V_1 = I_1 x_1 + (I_1 + I_2) x_2$

$$\text{i.e. } V_1 = 3I_1 + 2I_2 \quad \dots\dots\dots (i)$$

And the mesh equation for the closed mesh through 'cd' is given by: $0 = I_2 x_1$

$$+ (I_1 + I_2) x_2 \text{ or } 3I_2 + 2I_1 = 0 \text{ or}$$

$$I_1 = -(3/2) \cdot I_2 \quad \dots\dots\dots (ii)$$

Using equation (ii) in the equation (i) above we get :

$$V_1 = -(9/2) I_2 + 2I_2 = -(5/2)I_2$$

$$\text{Or } V_1 / -I_2 = B = (5/2)$$

And from equation (ii) above we can directly get

$$I_1 / -I_2 = D = 3/2$$

Hence the transmission parameters can be written in matrix notation as :

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4 & \frac{5}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$$

Here we can see that $AD - BC = 1$ and $A \neq D$

Hence the network is Symmetrical but not Reciprocal.

UNIT-III:

Locus diagrams:

- Resonance and Magnetic Circuits:
- Locus diagrams – Series and Parallel RL, RC, RLC circuits with variation of various parameters –
- Resonance-Series and Parallel circuits,
- Concept of band width and quality factor.
- Magnetic Circuits- Faraday's laws of electromagnetic induction,
- Concept of self and mutual inductance,
- Dot convention, Coefficient of coupling,
- Composite magnetic circuits,
- Analysis of series and parallel magnetic circuits.

Locus Diagrams with variation of various parameters:

Introduction: In AC electrical circuits the magnitude and phase of the current vector depends upon the values of R,L&C when the applied voltage and frequency are kept constant. The path traced by the terminus (tip) of the current vector when the parameters R,L&C are varied is called the current **Locus diagram**. Locus diagrams are useful in studying and understanding the behavior of the RLC circuits when one of these parameters is varied keeping voltage and frequency constant.

In this unit, Locus diagrams are developed and explained for series RC,RL circuits and Parallel LC circuits along with their internal resistances when the parameters R,L and C are varied.

The term circle diagram identifies locus plots that are either circular or semicircular. The defining equations of such circle diagrams are also derived in this unit for series RC and RL diagrams.

In both series RC,RL circuits and parallel LC circuits resistances are taken to be in series with L and C to highlight the fact that all practical L and C components will have at least a small value of internal resistance. Series RL circuit with varying Resistance R:

Refer to the series RL circuit shown in the figure (a) below with constant X_L and varying R. The current I_L lags behind the applied voltage V by a phase angle $\theta = \tan^{-1}(X_L/R)$ for a given value of R as shown in the figure (b) below. When $R=0$ we can see that the current is maximum equal to V/X_L and lies along the I axis with phase angle equal to 90° . When R is increased from zero to infinity the current gradually reduces from V/X_L to 0 and phase angle also reduces from 90° to 0° .

As can be seen from the figure, the tip of the current vector traces the path of a semicircle

With its diameter along the +ve I axis.

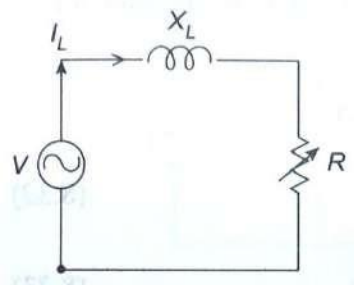


Fig 4.1(a): Series RL circuit with Varying Resistance R

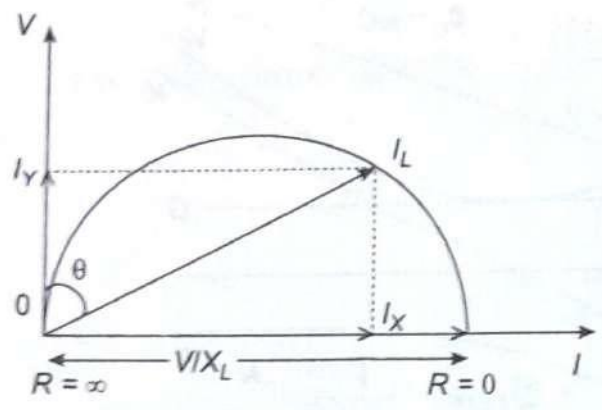


Fig 4.1(b): Locus of current vector I_L with variation of R

The related equations are:

$$I_L = V/Z \quad \sin \theta = X_L/Z \quad \text{or } Z = X_L / \sin \theta \quad \text{and } \cos \theta = R/Z$$

$$\text{Therefore } I_L = (V/X_L) \sin \theta$$

For constant V and X_L the above expression for I_L is the polar equation of a circle with diameter (V/X_L) as shown in the figure above.

Circle equation for the RL circuit: (with fixed reactance and variable Resistance):

The X and Y coordinates of the current I_L are $I_X = I_L \sin \theta$

$$I_Y = I_L \cos \theta$$

From the relations given above and earlier we get

$$I_X = (V/Z)(X_L/Z) = VX_L/Z^2 \quad \text{----- (1)}$$

$$\text{and} \quad I_Y = (V/Z)(R/Z) = VR/Z^2 \quad \text{----- (2)}$$

Squaring and adding the above two equations we get

$$I_X^2 + I_Y^2 = V^2(X_L^2 + R^2)/Z^4 = (V^2 Z^2)/Z^4 = V^2/Z^2 \quad \text{----- (3)}$$

From equation (1) above we have $Z^2 = VX_L/I_X$ and substituting this in the above equation (3) we get:

$$I_X^2 + I_Y^2 = V^2/(VX_L/I_X) = (V/X_L)I_X \quad \text{or}$$

$$I_X^2 + I_Y^2 - (V/X_L)I_X = 0$$

Adding $(V/2X_L)^2$ to both sides, the above equation can be written as

$$[I_X - V/2X_L]^2 + I_Y^2 = (V/2X_L)^2 \quad \text{----- (4)}$$

Equation (4) above represents a circle with a radius of $(V/2X_L)$ and with its coordinates of the centre as $(V/2X_L, 0)$

Series RC circuit with varying Resistance R:

Refer to the series RC circuit shown in the figure (a) below with constant X_C and varying R. The current I_C leads the

applied voltage V by a phase angle $\Theta = \tan^{-1}(X_C/R)$ for a given value of R as shown in the figure (b) below. When $R=0$ we can see that the current is maximum equal to $-V/X_C$ and lies along the negative I axis with phase angle equal to -90° . When R is increased from zero to infinity the current gradually reduces from $-V/X_C$ to 0 and phase angle also reduces from -90° to 0° . As can be seen from the figure, the tip of the current vector traces the path of a semicircle but now with its diameter along the negative I axis.

Circle equation for the RC circuit: (with fixed reactance and variable Resistance):

In the same way as we got for the Series RL circuit with varying resistance we can get the circle equation for an RC circuit with varying resistance as :

$$[I_X + V/2X_C]^2 + I_Y^2 = (V/2X_C)^2$$

whose coordinates of the centre are $(-V/2X_C, 0)$ and radius equal to $V/2X_C$

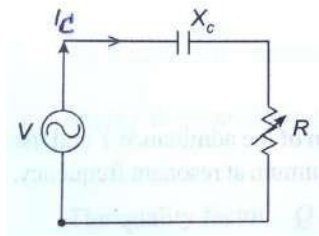


Fig 4.2 (a): Series RC circuit with Varying Resistance R

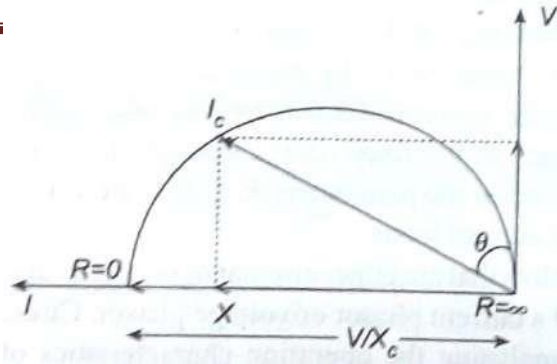


Fig 4.2 (b): Locus of current vector I_c with variation of R

Series RL circuit with varying Reactance X_L :

Refer to the series RL circuit shown in the figure (a) below with constant R and varying X_L . The current I_L lags behind the applied voltage V by a phase angle $\theta = \tan^{-1}(X_L/R)$ for a given value of R as shown in the figure (b) below. When $X_L = 0$ we can see that the current is maximum equal to V/R and lies along the +ve V axis with phase angle equal to 0° . When X_L is increased from zero to infinity the current gradually reduces from V/R to 0 and phase angle increases from 0° to 90° . As can be seen from the figure, the tip of the current vector traces the path of a semicircle with its diameter along the +ve V axis and on to its right side.

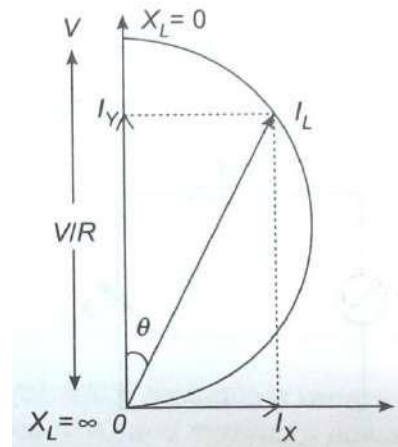
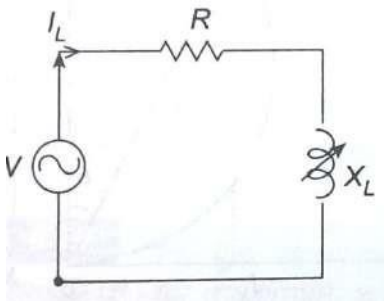


Fig 4.3(a): Series RL circuit with varying X_L Fig 4.3(b) : Locus of current vector I_L with variation of X_L

Series RC circuit with varying Reactance X_c :

Refer to the series RC circuit shown in the figure (a) below with constant R and varying X_c . The current I_c leads the applied voltage V by a phase angle $\theta = \tan^{-1}(X_c/R)$ for a given value of R as shown in the figure (b) below. When $X_c = 0$ we can see that the current is maximum equal to V/R and lies along the V axis with phase angle equal to 0° . When X_c is increased from zero to infinity the current gradually reduces from V/R to 0 and phase angle increases from 0° to -90° . As can be seen from the figure, the tip of the current vector traces the path of a semicircle with its diameter along the +ve V axis but now on to its left side.

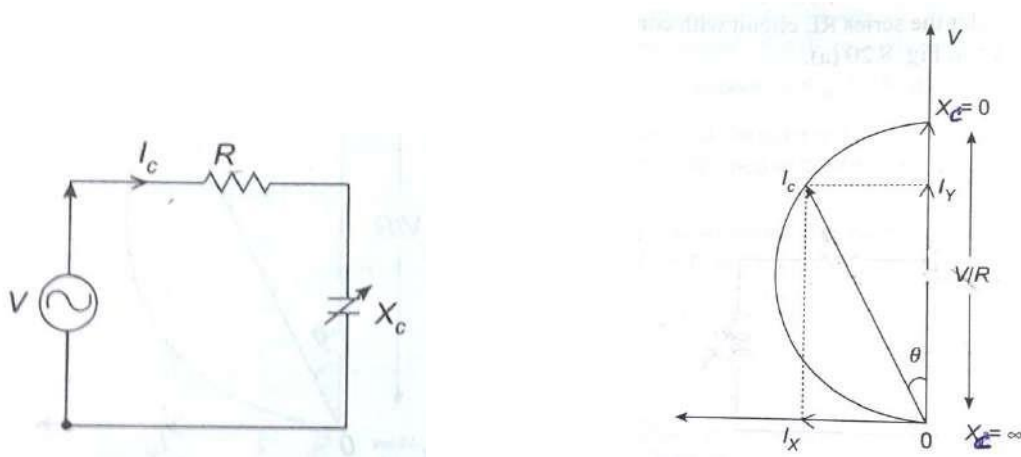
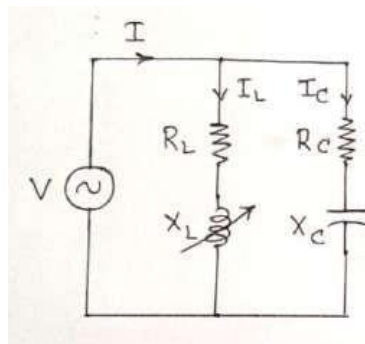


Fig4.4(a):Series RC circuit with varying X_c Fig4.4(b): Locus of current vector I_c with variation of X_c

Parallel LC circuits:

Parallel LC circuit along with its internal resistances as shown in the figures below is considered here for drawing the locus diagrams. As can be seen, there are two branch currents I_c and I_L along with the total current I . Locus diagrams of the current I_L or I_c (depending on which arm is varied) and the total current I are drawn by varying R_L , R_c , X_L and X_c one by one.

Varying X_L :



The current I_C through the capacitor is constant since R_C and C are fixed and it leads the voltage vector OV by an angle $\theta_C = \tan^{-1}(X_C/R_C)$ as shown in the figure (b). The current I_L through the inductance is the vector OL . Its amplitude is maximum and equal to V/R_L when X_L is zero and it is in phase with the applied voltage V . When X_L is increased from zero to infinity its amplitude decreases to zero and phase will be lagging the voltage by 90° . In between, the phase angle will be lagging the voltage V by an angle $\theta_L = \tan^{-1}(X_L/R_L)$. The locus of the current vector I_L is a semicircle with a diameter of length equal to V/R_L . Note that this is the same locus what we got earlier for the series RL circuit with X_L varying except that here V is shown horizontally.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors starting point (we will take varying amplitude vector I_L) at the tip of the other vector (we will take constant amplitude vector I_C) and then join the start of fixed vector I_C to the end of varying vector I_L . Using this principle we can get the locus of the total current vector OI by shifting the I_L semicircle starting point O to the end of current vector OI_C keeping the two diameters parallel. The resulting semi circle $I_C I_B$ shown in the figure in dotted lines is the locus of the total current vector OI .

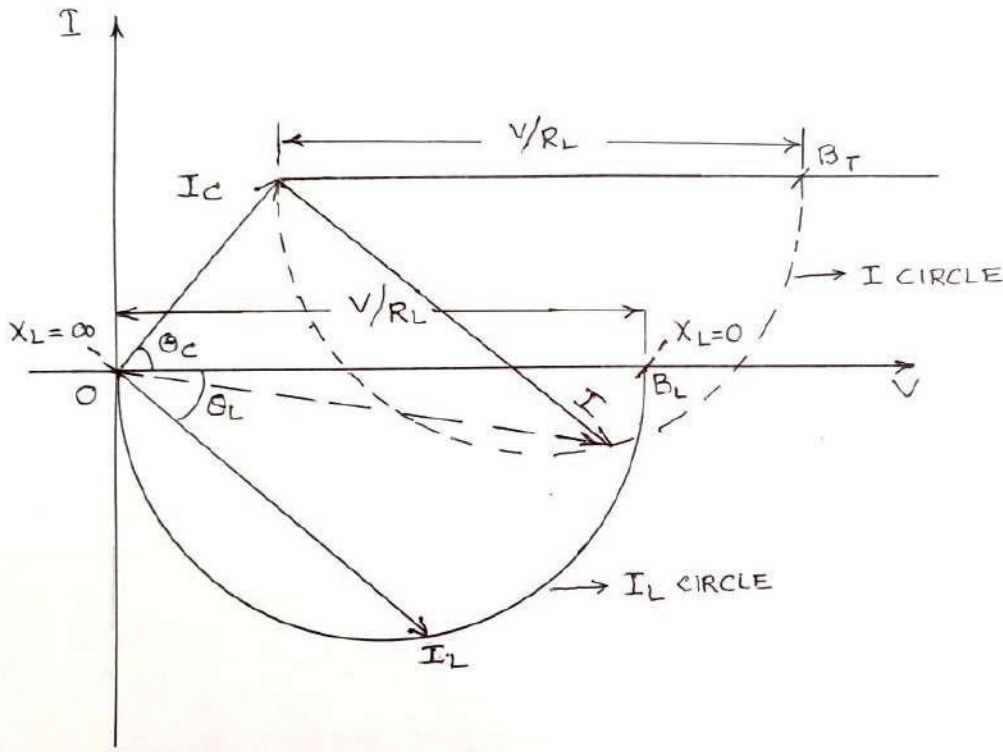


Fig 4.5(b): Locus of current vector I in Parallel LC circuit when X_L is varied from 0 to ∞

Varying X_C :

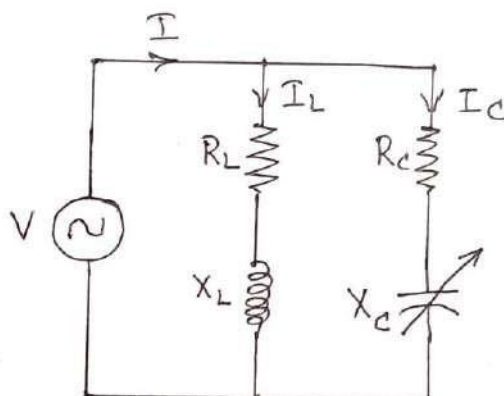


Fig.4.6(a) parallel LC circuit with Internal Resistances R_L and R_C in series with L (fixed) and C (Variable) respectively.

The current I_L through the inductor is constant since R_L and L are fixed and it lags the voltage vector OV by an angle $\Theta_L = \tan^{-1}(X_L/R_L)$ as shown in the figure (b). The current I_C through the capacitance is the vector OI_C . Its amplitude is maximum and equal to V/R_C when X_C is zero and it is in phase with the applied voltage V . When X_C is increased from zero to infinity its amplitude decreases to zero and phase will be leading the voltage by 90° . In between, the phase angle will be leading the voltage V by an angle $\Theta_C = \tan^{-1}(X_C/R_C)$. The locus of the current vector I_C is a semicircle with a diameter of length equal to V/R_C as shown in the figure below. Note that this is the same locus what we got earlier for the series RC circuit with X_C varying except that here V is shown horizontally.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors starting point (we will take varying amplitude vector I_C) at the tip of the other vector (we will take constant amplitude vector I_L) and then join the start of the fixed vector I_L to the end of varying vector I_C . Using this principle we can get the locus of the total current vector OI by shifting the I_C semicircle starting point O to the end of current vector OI_L keeping the two diameters parallel. The resulting semicircle $I_L I_B T$ shown in the figure in dotted lines is the locus of the total current vector OI .

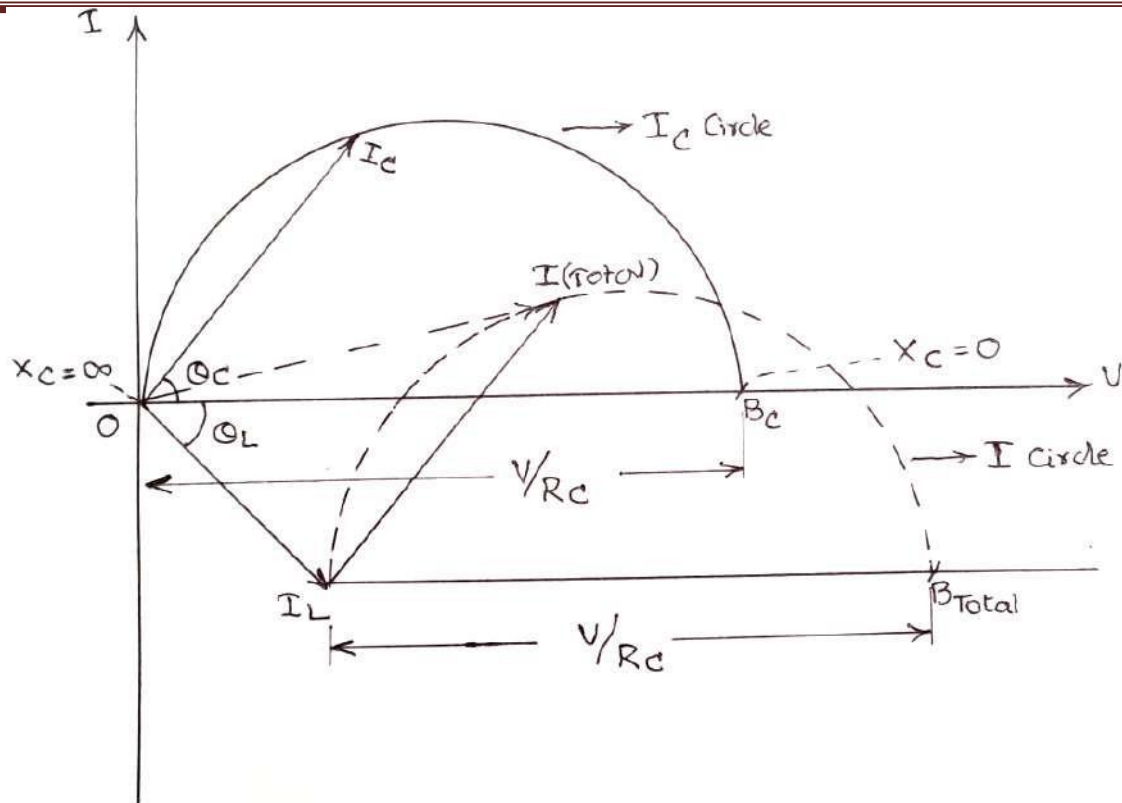


Fig4.6 (b) : Locus of current vector I in Parallel LC circuit when X_c is varied from 0 to ∞

Varying R_L :

The current I_c through the capacitor is constant since R_c and C are fixed and it leads the voltage vector OV by an angle $\theta_c = \tan^{-1}(X_c/R_c)$ as shown in the figure (b). The current I_L through the inductance is the vector OL . Its amplitude is maximum and equal to V/X_L when R_L is zero. Its phase will be lagging the voltage by 90° . When R_L is increased from zero to infinity its amplitude decreases to zero and it is in phase with the applied voltage V . In between, the phase angle will be lagging the voltage V by an angle $\theta_L = \tan^{-1}(X_L/R_L)$. The locus of the current vector I_L is a semicircle with a diameter of length equal to V/R_L . Note that this is the same locus what we got earlier for the series RL circuit with R varying except that here V is shown horizontally.

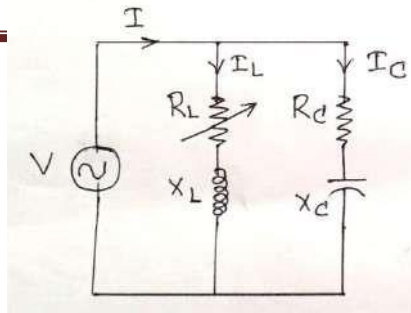


Fig.4.7(a) parallel LC circuit with Internal Resistances R_L (Variable) and R_C (fixed) in series with L and C respectively.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors starting point (we will take varying amplitude vector I_L) at the tip of the other vector (we will take constant amplitude vector I_C) and then join the start of fixed vector I_C to the end of varying vector I_L . Using this principle we can get the locus of the total current vector OI by shifting the I_L semicircle starting point O to the end of current vector OI_C keeping the two diameters parallel. The resulting semicircle $I_C I_B I$ shown in the figure in dotted lines is the locus of the total current vector OI .

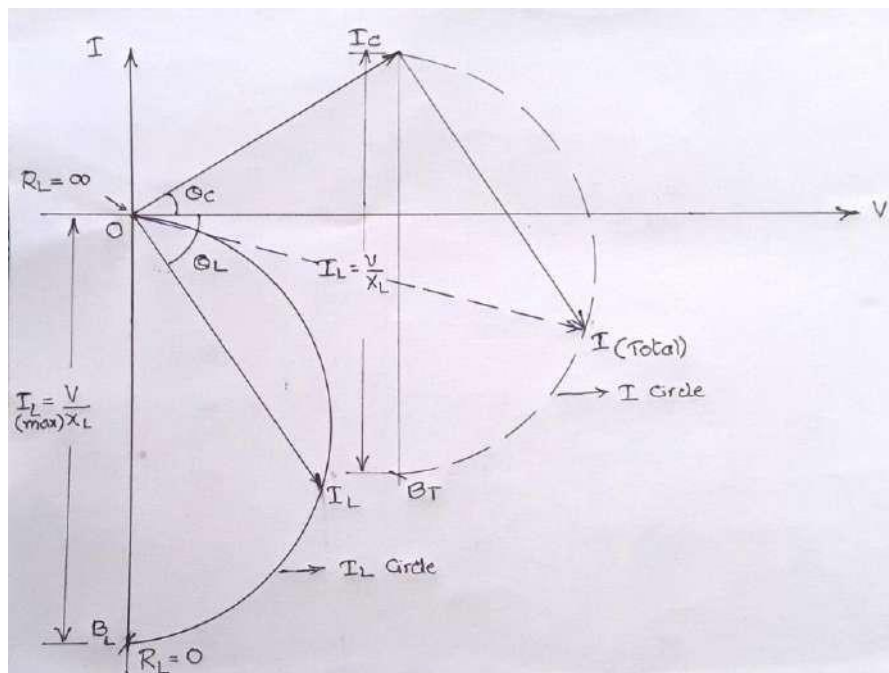


Fig 4.7(b) : Locus of current vector I in Parallel LC circuit when R_L is varied from 0 to ∞

Varying R_C :

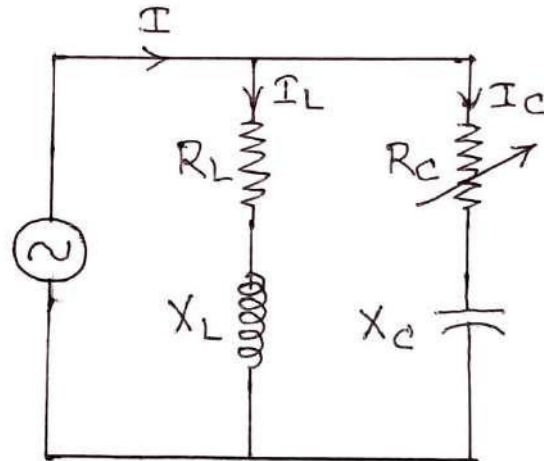


Fig. 4.8(a) parallel LC circuit with Internal Resistances R_L (fixed) and R_C (Variable) in series with L and C respectively.

The current I_L through the inductor is constant since R_L and L are fixed and it lags the voltage vector OV by an angle $\theta_L = \tan^{-1}(X_L/R_L)$ as shown in the figure (b). The current I_C through the capacitance is the vector OI_C . Its amplitude is maximum and equal to V/X_C when R_C is zero and its phase will be leading the voltage by 90° . When R_C is increased from zero to infinity its amplitude decreases to zero and it will be in phase with the applied voltage V. In between, the phase angle will be leading the voltage V by an angle $\theta_C = \tan^{-1}(X_C/R_C)$. The locus of the current vector I_C is a semicircle with a diameter of length equal to V/X_C as shown in the figure below. Note that this is the same locus what we got earlier for the series RC circuit with R varying except that here V is shown horizontally.

Now, to get the locus of the total current vector OI we have to add vectorially the currents I_C and I_L . We know that to get the sum of two vectors geometrically we have to place one of the vectors starting point (we will take varying amplitude vector I_C) at the tip of the other vector (we will take constant amplitude vector I_L) and then join the start of the fixed vector I_L to the end of varying vector I_C . Using this principle we can get the locus of the total current vector OI by shifting the I_C semicircle starting point O to the end of current vector OI_L keeping the two diameters parallel. The resulting semicircle $I_L I_B T$ shown in the figure in dotted lines is the locus of the total current vector OI.

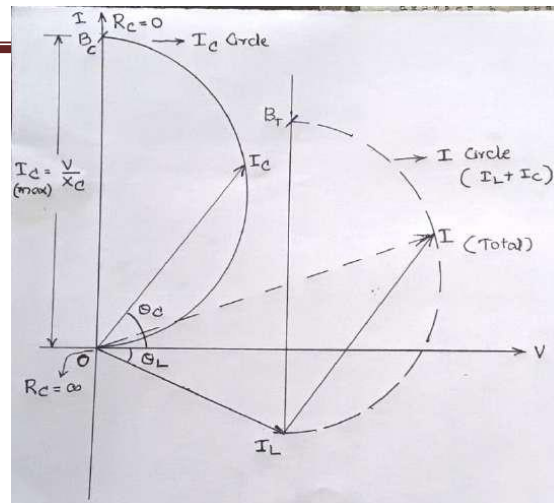


Fig 4.8(b) : Locus of current vector I in Parallel LC circuit when R_C is varied from 0 to ∞

Resonance:

Series RLC circuit:

The impedance of the series RLC circuit shown in the figure below and the current I through the circuit are given by :

$$Z = R + j\omega L + 1/j\omega C = R + j(\omega L - 1/\omega C) \quad I = V_s/Z$$

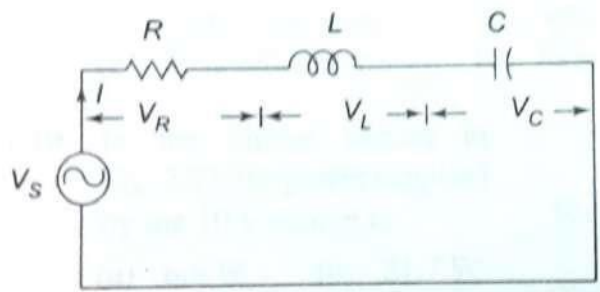


Fig 4.9: Series RLC circuit

The circuit is said to be in resonance when the Inductive reactance is equal to the Capacitive reactance. i.e. $X_L = X_C$ or $\omega L = 1/\omega C$. (i.e. Imaginary of the impedance is zero) The frequency at which the resonance occurs is called resonant frequency. In the resonant condition when $X_L = X_C$ they cancel with each other since they are in phase opposition (180° out of phase) and net impedance of the circuit is purely resistive. In this condition the magnitudes of voltages across

the Capacitance and the Inductance are also equal to each other but again since they are of opposite polarity they cancel with each other and the entire applied voltage appears across the Resistance alone.

Solving for the resonant frequency from the above condition of Resonance : $\omega L = 1/\omega C$

$$2\pi f_r L = 1/2\pi f_r C$$

$$f_r^2 = 1/4\pi^2 LC \quad \text{and} \quad f_r = 1/2\pi\sqrt{LC}$$

In a series RLC circuit, resonance may be produced by varying L or C at a fixed frequency or by varying frequency at fixed L and C.

Reactance, Impedance and Resistance of a Series RLC circuit as a function of frequency:

From the expressions for the Inductive and capacitive reactance we can see that when the frequency is zero, capacitance acts as an open circuit and Inductance as a short circuit. Similarly when the frequency is infinity inductance acts as an open circuit and the capacitance acts as a short circuit. The variation of Inductive and capacitive reactance along with Resistance R and the Total Impedance are shown plotted in the figure below.

As can be seen, when the frequency increases from zero to ∞ Inductive reactance X_L (directly proportional to ω) increases from zero to ∞ and capacitive reactance X_C (inversely proportional to ω) decreases from $-\infty$ to zero. Whereas, the Impedance decreases from ∞ to Pure Resistance R as the frequency increases from zero to f_r (as capacitive reactance reduces from $-\infty$ and becomes equal to Inductive reactance) and then increases from R to ∞ as the frequency increases from f_r to ∞ (as inductive reactance increases from its value at resonant frequency to ∞)

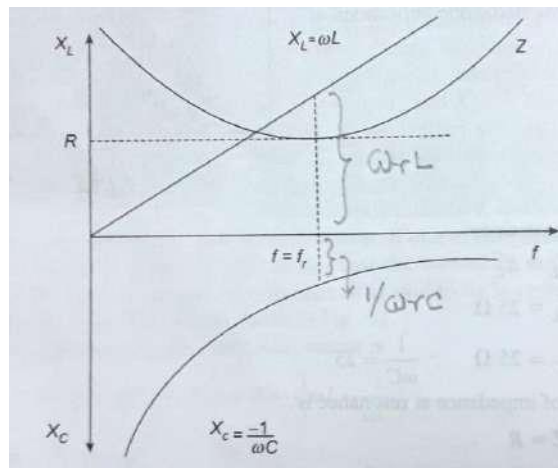


Fig 4.10: Reactance and Impedance plots of a Series RLC circuit

Phase angle of a Series RLC circuit as a function of frequency:

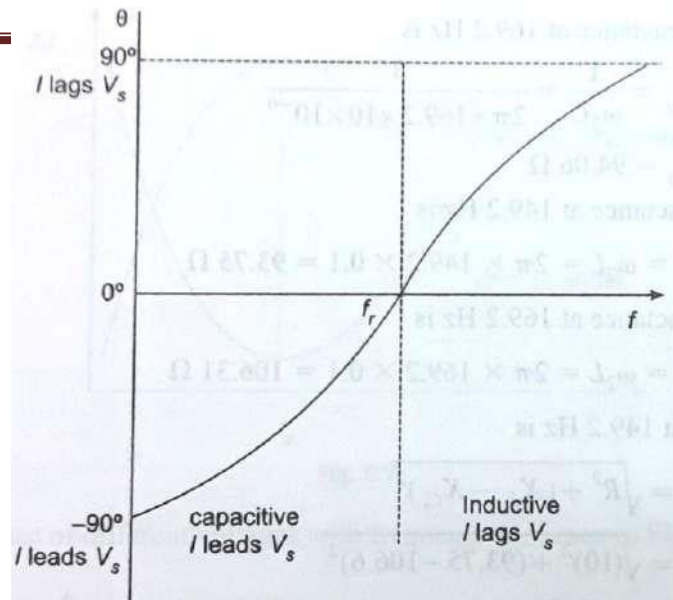


Fig4.11 : Phase plot of a Series RLC circuit

The following points can be seen from the Phase angle plot shown in the figure above:

- At frequencies below the resonant frequency capacitive reactance is higher than the inductive reactance and hence the phase angle of the current leads the voltage.
- As frequency increases from zero to f_r the phase angle changes from -90° to zero.
- At frequencies above the resonant frequency inductive reactance is higher than the capacitive reactance and hence the phase angle of the current lags the voltage.
- As frequency increases from f_r and approaches ∞ , the phase angle increases from zero and approaches 90°

Band width of a Series RLC circuit:

The band width of a circuit is defined as the Range of frequencies between which the output power is half of or 3 db less than the output power at the resonant frequency. These frequencies are called the cutoff frequencies, 3db points or half power points. But when we consider the output voltage or current, the range of frequencies between which the output voltage or current falls to 0.707 times of the value at the resonant frequency is called the Bandwidth BW. This is because voltage/current are related to power by a factor of $\sqrt{2}$ and when we are consider $\sqrt{2}$ times less it becomes 0.707. But still these frequencies are called as cutoff frequencies, 3db points or half power points. The lower end frequency is called **lower cutoff frequency** and the higher end frequency is called upper **cutoff frequency**.

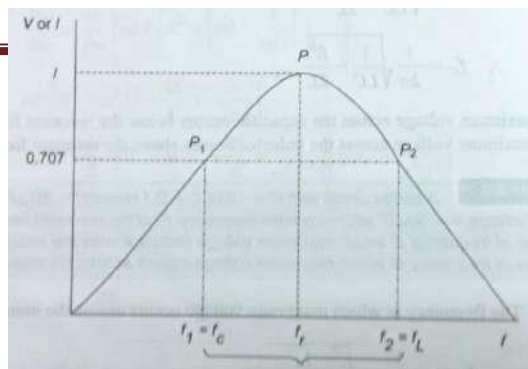


Fig 4.12: Plot showing the cutoff frequencies and Bandwidth of a series RLC circuit

Derivation of an expression for the BW of a series RLC circuit:

We know that $BW = f_2 - f_1$ Hz

If the current at points P_1 and P_2 are $0.707 (1/\sqrt{2})$ times that of I_{\max} (current at the resonant frequency) then the Impedance of the circuit at points P_1 and P_2 is $\sqrt{2}R$ (i.e. $\sqrt{2}$ times the impedance at f_r)

But Impedance at point P_1 is given by: $Z = \sqrt{R^2 + (1/\omega_1 C - \omega_1 L)^2}$ and equating this to $\sqrt{2}R$ we get :

$$(1/\omega_1 C) - \omega_1 L = R \quad \text{-----} \quad (1)$$

Similarly Impedance at point P_2 is given by: $Z = \sqrt{R^2 + (\omega_2 L - 1/\omega_2 C)^2}$ and equating this to $\sqrt{2}R$ we get:

$$\omega_2 L - (1/\omega_2 C) = R \quad \text{-----} \quad (2)$$

Equating the above equations (1) and (2) we get:

$$1/\omega_1 C - \omega_1 L = \omega_2 L - 1/\omega_2 C$$

Rearranging we get $L(\omega_1 + \omega_2) = 1/C[(\omega_1 + \omega_2)/\omega_1 \omega_2]$ i.e $\omega_1 \omega_2 = 1/LC$

But we already know that for a series RLC circuit the resonant frequency is given by $\omega^2 = 1/LC$ Therefore:

$$\omega_1 \omega_2 = \omega_r^2 \quad \text{----} \quad (3) \quad \text{and} \quad 1/C = \omega_r^2 L \quad \text{-----} \quad (4)$$

Next adding the above equations (1) and (2) we get:

$$1/\omega_1 C - \omega_1 L + \omega_2 L - 1/\omega_2 C = 2R$$

$$(\omega_2 - \omega_1)L + (1/\omega_1 C - 1/\omega_2 C) = 2R$$

$$(\omega_2 - \omega_1)L + 1/C[(\omega_2 - \omega_1)/\omega_1 \omega_2] = 2R \quad \text{-----} \quad (5)$$

Using the values of $\omega_1 \omega_2$ and $1/C$ from equations (3) and (4) above into equation (5) above we get:

$$(\omega_2 - \omega_1)L + \omega_r^2 L[(\omega_2 - \omega_1)/\omega_r^2] = 2R$$

$$\text{B.W. (ECE)} \quad 2L(\omega_2 - \omega_1) = 2R \quad \text{i.e. } (\omega_2 - \omega_1) = R/L \quad \text{and} \quad (f_2 - f_1) = R/2\pi L \quad \text{---R-18(6)}$$

$$\text{Or finally Bandwidth} \quad \mathbf{BW = R/2\pi L} \quad \text{---- (7)}$$

Since f_r lies in the centre of the lower and upper cutoff frequencies f_1 and f_2 using the above equation (6) we can get:

$$\mathbf{f_1 = f_r - R/4\pi L} \quad \text{----- (8)}$$

$$\mathbf{f_2 = f_r + R/4\pi L} \quad \text{----- (9)}$$

Further by dividing the equation (6) above by f_r on both sides we get another **important** relation: $(f_2 - f_1) / f_r = R/2\pi f_r L$ or $\mathbf{BW / f_r = R/2\pi f_r L}$ ----- (10)

Here an important property of a coil i.e. **Q factor** or **figure of merit** is defined as the ratio of the reactance to the resistance of a coil.

$$\mathbf{Q = 2\pi f_r L / R} \quad \text{----- (11)}$$

Now using the relation (11) we can rewrite the relation (10) as

$$\mathbf{Q = f_r / BW} \quad \text{----- (12)}$$

Quality factor of a series RLC circuit:

The quality factor of a series RLC circuit is defined as:

$$\mathbf{Q = \text{Reactive power in Inductor (or Capacitor) at resonance} / \text{Average power at Resonance}}$$

$$\text{Reactive power in Inductor at resonance} = I^2 X_L$$

$$\text{Reactive power in Capacitor at resonance} = I^2 X_C$$

$$\text{Average power at Resonance} = I^2 R$$

Here the power is expressed in the form $I^2 X$ (not as V^2/X) since I is common through R, L and C in the series RLC circuit and it gets cancelled during the simplification.

$$\text{Therefore } \mathbf{Q = I^2 X_L / I^2 R = I^2 X_C / I^2 R}$$

$$\text{i.e. } \mathbf{Q = X_L / R = \omega_r L / R} \quad \text{----- (1)}$$

$$\text{Or } \mathbf{Q = X_C / R = 1/\omega_r RC} \quad \text{----- (2)}$$

From these two relations we can also define Q factor as :

Substituting the value of $\omega_r = 1/\sqrt{LC}$ in the expressions (1) or (2) for **Q** above we can get the value of **Q** in terms of **R, L, C** as below.

$$Q = (1/\sqrt{LC}) L / R = (1/R) (\sqrt{L/C})$$

Selectivity:

Selectivity of a series **RLC** circuit indicates how well the given circuit responds to a given resonant frequency and how well it rejects all other frequencies. i.e. the selectivity is directly proportional to **Q** factor. A circuit with a good selectivity (or a high **Q** factor) will have maximum gain at the resonant frequency and will have minimum gain at other frequencies i.e. it will have very low band width. This is illustrated in the figure below.

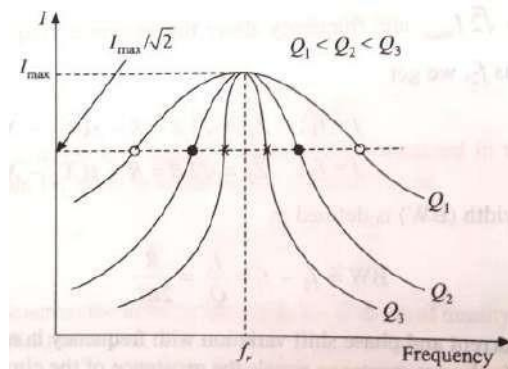


Fig 4.13: Effect of quality factor on bandwidth Voltage Magnification at resonance:

At resonance the voltages across the Inductance and capacitance are much larger than the applied voltage in a series RLC circuit and this is called voltage magnification at Resonance. The voltage magnification is equal to the **Q** factor of the circuit. This is proven below.

If we take the voltage applied to the circuit as **V** and the current through the circuit at resonance as **I** then

The voltage across the inductance **L** is: $V_L = IX_L = (V/R) \omega_r L$ and

The voltage across the capacitance **C** is: $V_C = IX_C = V/R \omega_r C$

But we know that the **Q** of a series RLC circuit = $\omega_r L / R = 1/R \omega_r C$ Using these relations in the expressions for **V_L** and **V_C** given above we get **V_L = VQ** and **V_C = VQ**

The ratio of voltage across the Inductor or capacitor at resonance to the applied voltage in a series RLC circuit is called Voltage magnification and is given by

$$\text{Magnification} = Q = V_L/V \quad \text{or } V_c / V$$

Important points In Series RLC circuit at resonant frequency :

- ☐ The impedance of the circuit becomes purely resistive and minimum i.e $Z = R$
- ☐ The current in the circuit becomes maximum
- ☐ The magnitudes of the capacitive Reactance and Inductive Reactance become equal
- ☐ The voltage across the Capacitor becomes equal to the voltage across the Inductor at resonance and is Q times higher than the voltage across the resistor

Bandwidth and Q factor of a Parallel RLC circuit:

Parallel RLC circuit is shown in the figure below. For finding out the **BW** and **Q** factor of a parallel RLC circuit, since it is easier we will work with Admittance , Conductance and Susceptance instead of Impedance, Resistance and Reactance like in series RLC circuit.

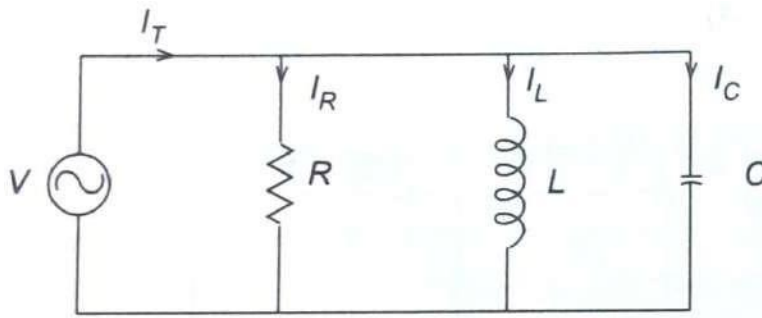


Fig 4.14 : Parallel RLC circuit

Then we have the relation:

$$Y = 1/Z = 1/R + 1/j\omega L + j\omega C = 1/R + j(\omega C - 1/\omega L)$$

For the parallel RLC circuit also, at resonance, the imaginary part of the Admittance is zero and hence the frequency at which resonance occurs is given by: $\omega_r C - 1/\omega_r L = 0$. From this we get: $\omega_r C = 1/\omega_r L$ and $\omega_r = 1/\sqrt{LC}$

which is the same value for ω_r as what we got for the series RLC circuit.

At resonance when the imaginary part of the admittance is zero the **admittance** becomes **minimum**. (i.e **Impedance** becomes **maximum** as against Impedance becoming minimum in series RLC circuit) i.e. Current becomes minimum in the parallel RLC circuit at resonance (as against current becoming maximum in series RLC circuit) and increases on either side of the resonant frequency as shown in the figure below.

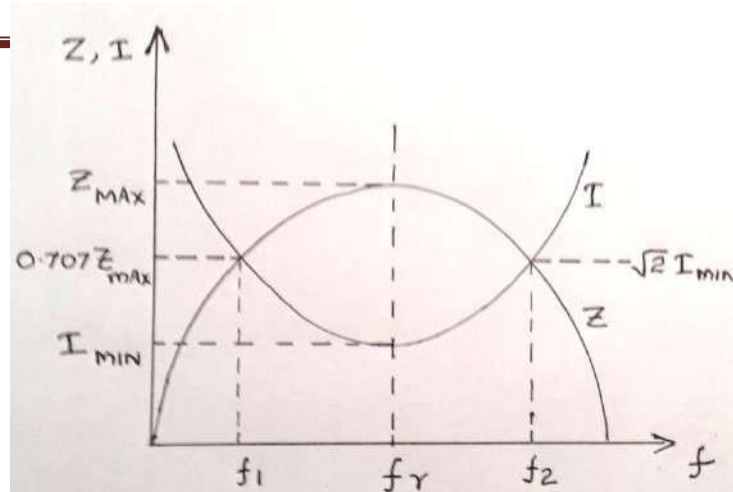


Fig 4.15: Variation of Impedance and Current with frequency in a Parallel RLC circuit

Here also the BW of the circuit is given by $BW = f_2 - f_1$ where f_2 and f_1 are still called the upper and lower cut off frequencies but they are 3db higher cutoff frequencies since we notice that at these cutoff frequencies the amplitude of the current is $\sqrt{2}$ times higher than that of the amplitude of current at the resonant frequency.

The BW is computed here also on the same lines as we did for the series RLC circuit:

If the current at points P_1 and P_2 is $\sqrt{2}$ (3db) times higher than that of I_{min} (current at the resonant frequency) then the admittance of the circuit at points P_1 and P_2 is also $\sqrt{2}$ times higher than the admittance at f_r

But amplitude of admittance at point P_1 is given by: $Y = \sqrt{1/R^2 + (\omega_1 L - \omega_1 C)^2}$ and equating this to $\sqrt{2}/R$ we get

$$1/\omega_1 L - \omega_1 C = 1/R \quad \text{----- (1)}$$

Similarly amplitude of admittance at point P_2 is given by: $Y = \sqrt{1/R^2 + (\omega_2 C - 1/\omega_2 L)^2}$ and equating this to $\sqrt{2}/R$ we get

$$\omega_2 C - 1/\omega_2 L = 1/R \quad \text{----- (2)}$$

Equating LHS of (1) and (2) and further simplifying we get

$$1/\omega_1 L - \omega_1 C = \omega_2 C - 1/\omega_2 L$$

$$1/\omega_1 L + 1/\omega_2 L = \omega_1 C + \omega_2 C$$

$$1/L [(\omega_1 + \omega_2)/\omega_1 \omega_2] = (\omega_1 + \omega_2)C$$

$$1/L C = \omega_1 \omega_2$$

Next adding the equations (1) and (2) above and further simplifying we get

$$1/\omega_1 L - \omega_1 C + \omega_2 C - 1/\omega_2 L = 2/R$$

$$(\omega_2 C - \omega_1 C) + (1/\omega_1 L - 1/\omega_2 L) = 2/R$$

$$(\omega_2 - \omega_1)C + 1/L[(\omega_2 - \omega_1)/\omega_1 \omega_2] = 2/R$$

Substituting the value of $\omega_1 \omega_2 = 1/LC$

$$(\omega_2 - \omega_1)C + LC/L[(\omega_2 - \omega_1)] = 2/R$$

$$(\omega_2 - \omega_1)C + C[(\omega_2 - \omega_1)] = 2/R \quad 2C$$

$$[(\omega_2 - \omega_1)] = 2/R$$

$$\text{Or } [(\omega_2 - \omega_1)] = 1/RC$$

From which we get the band width **BW = $f_2 - f_1 = 1/2\pi RC$**

Dividing both sides by f_r we get: **$(f_2 - f_1)/f_r = 1/2\pi f_r RC$ -----(1)**

Quality factor of a Parallel RLC circuit:

The quality factor of a Parallel RLC circuit is defined as:

Q = Reactive power in Inductor (or Capacitor) at resonance / Average power at Resonance

Reactive power in Inductor at resonance = V^2/X_L

Reactive power in Capacitor at resonance = V^2/X_C

Average power at Resonance = V^2/R

Here the power is expressed in the form V^2/X (not as $I^2 X$ as in series circuit) since V is common across R, L and C in the parallel RLC circuit and it gets cancelled during the simplification.

Therefore **$Q = (V^2/X_L) / (V^2/R) = (V^2/X_C) / (V^2/R)$**

i.e. **$Q = R/X_L = R/\omega_r L$** ----- (1)

Or **$Q = R/X_C = \omega_r RC$** ----- (2)

From these two relations we can also define **Q** factor as :

$$Q = \text{Resistance / Inductive (or Capacitive) reactance at resonance}$$

Substituting the value of $\omega_r = 1/\sqrt{LC}$ in the expressions (1) or (2) for Q above we can get the value of Q in terms of R, L, C as below.

$$Q = (1/\sqrt{LC}) RC = R(\sqrt{C}/L)$$

Further using the relation $Q = \omega_r RC$ (equation 2 above) in the earlier equation (1) we get in BW viz. $(f_2 - f_1)/f_r = 1/2\pi f_r RC$ we get:

$$(f_2 - f_1)/f_r = 1/Q \text{ or } Q = f_r / (f_2 - f_1) = f_r / BW$$

i.e. In Parallel RLC circuit also the Q factor is inversely proportional to the BW.

Admittance, Conductance and Susceptance curves for a Parallel RLC circuit as a function of frequency :

- The effect of varying the frequency on the Admittance, Conductance and Susceptance of a parallel circuit is shown in the figure below.
- Inductive susceptance B_L is given by $B_L = -1/\omega L$. It is inversely proportional to the frequency ω and is shown in the fourth quadrant since it is negative.
- Capacitive susceptance B_C is given by $B_C = \omega C$. It is directly proportional to the frequency ω and is shown in the first quadrant as OP . It is positive and linear.
- Net susceptance $B = B_C - B_L$ and is represented by the curve JK . As can be seen it is zero at the resonant frequency f_r .
- The conductance $G = 1/R$ and is constant.
- The total admittance Y and the total current I are minimum at the resonant frequency as shown by the curve VW .

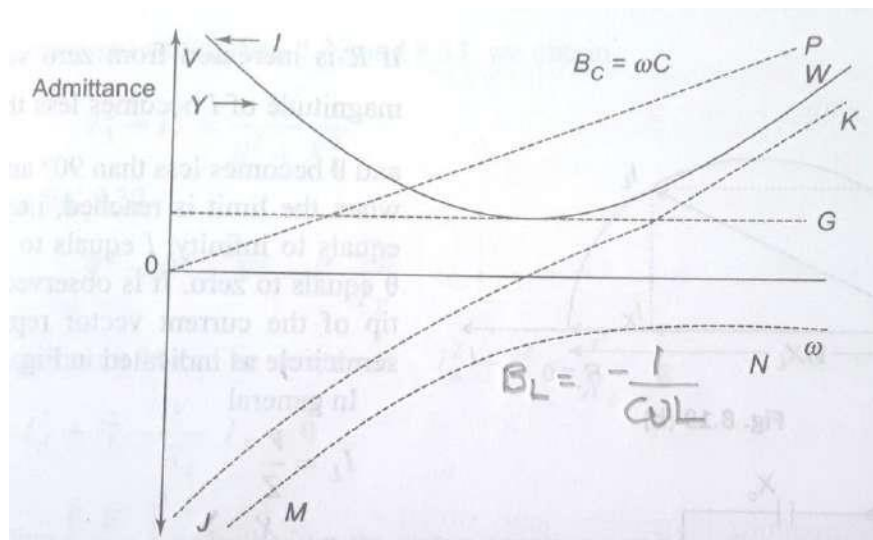


Fig4.16:Conductance,SusceptanceandAdmittanceplotsofaParallelRLCcircuit Current

magnification in a Parallel RLC circuit:

Just as voltage magnification takes place across the capacitance and Inductance at the resonant frequency in a series RLC circuit, current magnification takes place in the currents through the capacitance and Inductance at the resonant frequency in a Parallel RLC circuit. This is shown below.

Voltage across the Resistance = $V = IR$

Current through the Inductance at resonance $I_L = V/\omega_r L = IR/\omega_r L = I \cdot R/\omega_r L = IQ$ Similarly

Current through the Capacitance at resonance $I_C = V/(1/\omega_r C) = IR/(1/\omega_r C) = I(R\omega_r C) = IQ$

From which we notice that the quality factor $Q = I_L/I$ or I_C/I and that the current through the inductance and the capacitance increases by Q times that of the current through the resistor at resonance. .

Important points In Parallel RLC circuit at resonant frequency :

- The impedance of the circuit becomes resistive and maximum i.e $Z = R$
- The current in the circuit becomes minimum
- The magnitudes of the capacitive Reactance and Inductive Reactance become equal
- The current through the Capacitor becomes equal and opposite to the current through the Inductor at resonance and is Q times higher than the current through the resistor

Magnetic Circuits:

Introduction to the Magnetic Field:

Magnetic fields are the fundamental medium through which energy is converted from one form to another in motors, generators and transformers. Four basic principles describe how magnetic fields are used in these devices.

0. A current-carrying conductor produces a magnetic field in the area around it.

Explained in Detail by Fleming's Right hand rule and Amperes Law.

1. A time varying magnetic flux induces a voltage in a coil of wire if it passes through that coil. (basis of Transformer action)

Explained in detail by the Faradays laws of Electromagnetic Induction.

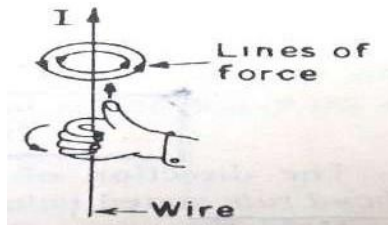
2. A current carrying conductor in the presence of a magnetic field has a force induced in it (Basis of Motor action)
3. A moving wire in the presence of a magnetic field has a voltage induced in it (Basis of Generator action)

We will be studying in this unit the first two principles in detail and the other two principles in the next unit on DC machines.

Two basic laws governing the production of a magnetic field by a current carrying conductor:

The direction of the magnetic field produced by a current carrying conductor is given by the **Flemings Right hand rule** and its' amplitude is given by the **Ampere's Law**.

Flemings right hand rule: Hold the conductor carrying the current in your right hand such that the Thumb points along the wire in the direction of the flow of current, then the fingers will encircle the wire along the lines of the Magnetic force.



Ampere's Law: The line integral of the magnetic field intensity H around a closed magnetic path is equal to the total current enclosed by the path.

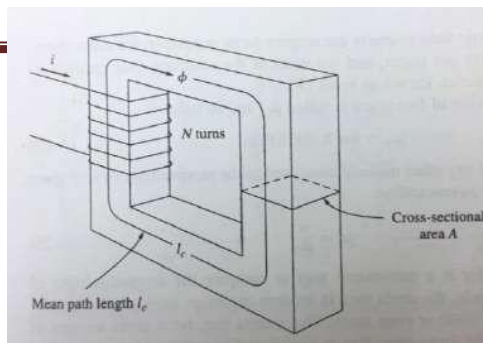
This is the basic law which gives the relationship between the Magnetic field Intensity H and the current I and is mathematically expressed as

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{net}}$$

where \mathbf{H} is the magnetic field intensity produced by the current I_{net} and $d\mathbf{l}$ is a differential element of length along the path of integration. \mathbf{H} is measured in *Ampere-turns per meter*.

Important parameters and their relation in magnetic circuits :

- Consider a current carrying conductor wrapped around a ferromagnetic core as shown in the figure below .



- Applying Ampere's law, the total amount of magnetic field induced will be proportional to the amount of current flowing through the conductor wound with N turns around the ferromagnetic material as shown. Since the core is made of ferromagnetic material, it is assumed that a majority of the magnetic field will be confined to the core.
- The path of integration in this case as per the Ampere's law is the mean path length of the core, l_c . The current passing within the path of integration I_{net} is then Ni , since the coil of wire cuts the path of integration N times while carrying the current i . Hence Ampere's Law becomes: **$Hl_c = Ni$**

Therefore

$$H = Ni/l_c$$

- In this sense, H (Ampere turns per meter) is known as the effort required to induce a magnetic field. The strength of the magnetic field flux produced in the core also depends on the material of the core. Thus: **$B = \mu H$** where

B = magnetic flux density [webers per square meter, or Tesla (T)]

μ = magnetic permeability of material (Henrys per meter)

H = magnetic field intensity (ampere-turns per meter)

- The constant μ may be further expanded to include **relative permeability** which can be defined as below:

$$\mu_r = \mu / \mu_0$$

where μ_0 = permeability of free space (equal to that of air)

- Hence the permeability value is a combination of the relative permeability and the permeability of free space. The value of relative permeability is dependent upon the type of material used. The higher the amount permeability, the higher the amount of flux induced in the core. Relative permeability is a convenient way to compare the magnetizability of materials.
- Also, because the permeability of iron is so much higher than that of air, the majority of the flux in an iron core remains inside the core instead of travelling through the surrounding air, which has lower permeability. The small leakage flux that does leave the iron core is important in

determining the flux linkages between coils and the self-inductances of coils in transformers and motors.

- In a core such as shown in the figure above

$$B = \mu H = \mu Ni / l_c$$

Now, to measure the total flux flowing in the ferromagnetic core, consideration has to be made in terms of its cross sectional area (CSA). Therefore:

$$\Phi = B \cdot dA \text{ where: } A = \text{cross sectional area throughout the core.}$$

Assuming that the flux density in the ferromagnetic core is constant throughout hence the equation simplifies to:

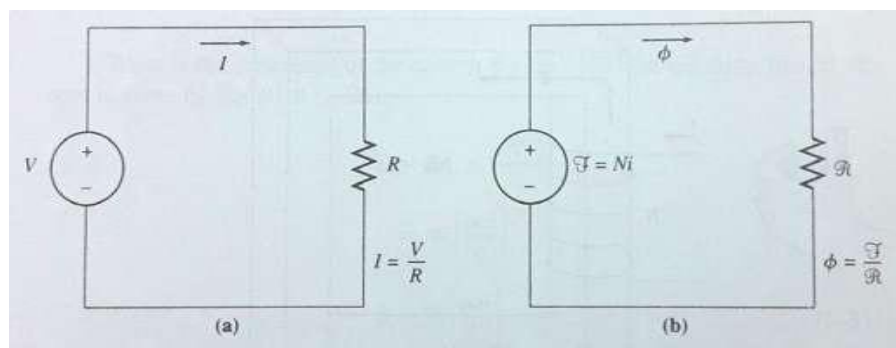
$$\Phi = B \cdot A$$

Taking the previous expression for **B** we get $\Phi = \mu NiA / l_c$

Electrical analogy of magnetic circuits:

The flow of magnetic flux induced in the ferromagnetic core is analogous to the flow of electric current in an electrical circuit hence the name magnetic circuit.

The analogy is as follows:



(a) Electric Circuit

(b) Electrical Analogy of Magnetic Circuit

Referring to the magnetic circuit analogy, **F** is denoted as **magnetomotive force** (mmf) which is similar to Electromotive force in an electrical circuit (emf). Therefore, we can say that **F** is the force which pushes magnetic flux around a ferromagnetic core with a value of **Ni** (refer to ampere's law). Hence **F** is measured in ampere turns. Hence the magnetic circuit equivalent equation is as shown:

$$F = \Phi \cdot R \text{ (similar to } V=IR)$$

We already have the relation $\Phi = \mu NiA/l$ and using this we get $R = F / \Phi = Ni / \Phi$

$$R = Ni / (\mu NiA/l) = l / \mu A$$

- The polarity of the mmf will determine the direction of flux. To easily determine the direction of flux, the '**right hand curl**' rule is applied:

When the direction of the curled fingers indicates the direction of current flow the resulting thumb direction will show the magnetic flux flow.

- The element of **R** in the magnetic circuit analogy is similar in concept to the electrical resistance. It is basically the measure of material resistance to the flow of magnetic flux. **Reluctance** in this analogy obeys the rule of electrical resistance (Series and Parallel Rules). Reluctance is measured in **Ampere-turns per weber**.
- The inverse of electrical resistance is conductance which is a measure of conductivity of a material. Similarly the inverse of reluctance is known as **permeance P** which represents the degree to which the material permits the flow of magnetic flux.
- By using the magnetic circuit approach, calculations related to the magnetic field in a ferromagnetic material are simplified but with a little inaccuracy.

Equivalent Reluctance of a series Magnetic circuit:

$$R_{\text{eqseries}} = R_1 + R_2 + R_3 + \dots$$

Equivalent Reluctance of a Parallel Magnetic circuit: $1/R_{\text{eqparallel}} = 1/R_1 + 1/R_2 + 1/R_3 + \dots$

Electromagnetic Induction and Faraday's law – Induced Voltage from a Time-Changing Magnetic Field:

Faraday's Law:

Whenever a varying magnetic flux passes through a turn of a coil of wire, voltage will be induced in the turn of the wire that is directly proportional to the rate of change of the flux linkage with the turn of the coil of wire.

$$e_{ind} \propto -d\phi/dt$$

$$e_{\text{ind}} = -k \cdot d\phi/dt$$

The negative sign in the equation above is in accordance to **Lenz' Law** which states:

The direction of the induced voltage in the turn of the coil is such that if the coil is short circuited, it would produce a current that would cause a flux which opposes the original change of flux.

And **k** is the constant of proportionality whose value depends on the system of units chosen. In the SI system of units **k=1** and the above equation becomes:

$$e_{\text{ind}} = - d\phi/dt$$

Normally a coil is used with several turns and if there are **N** number of turns in the coil with the same amount of flux flowing through it then:

$$e_{\text{ind}} = - N d\phi/dt$$

*Change in the flux linkage **Nϕ** of a coil can be obtained in two ways:*

1. Coil remains stationary and flux changes with time (Due to AC current like in Transformers and this is called Statically induced e.m.f)
2. Magnetic flux remains constant and stationary in space, but the coil moves relative to the magnetic field so as to create a change in the flux linkage of the coil (Like in Rotating machines and this is called Dynamically induced e.m.f.

Self inductance:

From the Faradays laws of Electromagnetic Induction we have seen that an e.m.f will be induced in a conductor when a time varying flux is linked with a conductor and the amplitude of the induced e.m.f is proportional to the rate of change of the varying flux.

If the time varying flux is produced by a coil of **N** turns then the coil itself links with the time varying flux produced by itself and an emf will be induced in the same coil. This is called self inductance .

The flux **ϕ** produced by a coil of **N** turns links with its own **N** turns of the coil and hence the total flux linkage is equal to **Nϕ = (μ N² A / l) I** [using the expression $\Phi = \mu NiA/l$ we already developed] Thus we see that the total magnetic flux produced by a coil of **N** turns and linked with itself is proportional to the current flowing through the coil i.e.

$$N\phi \propto I \text{ or } N\phi = L I$$

From the Faradays law of electromagnetic Induction, the self induced e.m.f for this coil of **N** turns is given by:

$$e_{\text{ind}} = -N \frac{d\Phi}{dt} = -L \frac{di}{dt}$$

The constant of proportionality L is called the self Inductance of the coil or simply Inductance and its value is given by $L = (\mu N^2 A / l)$. If the radius of the coil is r then:

$$L = (\mu N^2 \pi r^2 / l) i$$

From the above two equations we can see that Self Inductance of a coil can be defined as the flux produced per unit current i.e *Weber/Ampere* (equation1) or the induced emf per unit rate of change of current i.e *Volt-sec/Ampere* (equation 2)

The unit of Inductance is named after Joseph Henry as **Henry** and is given to these two combinations as :

$$1H = 1WbA^{-1} = 1VsA^{-1}$$

Self Inductance of a coil is defined as one Henry if an induced emf of one volt is generated when the current in the coil changes at the rate of one Ampere per second.

Henry is relatively a very big unit of Inductance and we normally use Inductors of the size of mH ($10^{-3} H$) or μH ($10^{-6} H$)

Mutual inductance and Coefficient of coupling:

In the case of Self Inductance an emf is induced in the same coil which produces the varying magnetic field. The same phenomenon of Induction will be extended to a separate second coil if it is located in the vicinity of the varying magnetic field produced by the first coil. Faradays law of electromagnetic Induction is equally applicable to the second coil also. A current flowing in one coil establishes a magnetic flux about that coil and also about a second coil nearby but of course with a lesser intensity. The time-varying flux produced by the first coil and surrounding the second coil produces a voltage across the terminals of the second coil. This voltage is proportional to the time rate of change of the current flowing through the first coil.

Figure (a) shows a simple model of two coils L_1 and L_2 , sufficiently close together that the flux produced by a current $i_1(t)$ flowing through L_1 establishes an open-circuit voltage $v_2(t)$ across the terminals of L_2 . **Mutual inductance, M_{21}** , is defined such that

$$v_2(t) = M_{21} di_1(t)/dt \quad \text{-----[1]}$$

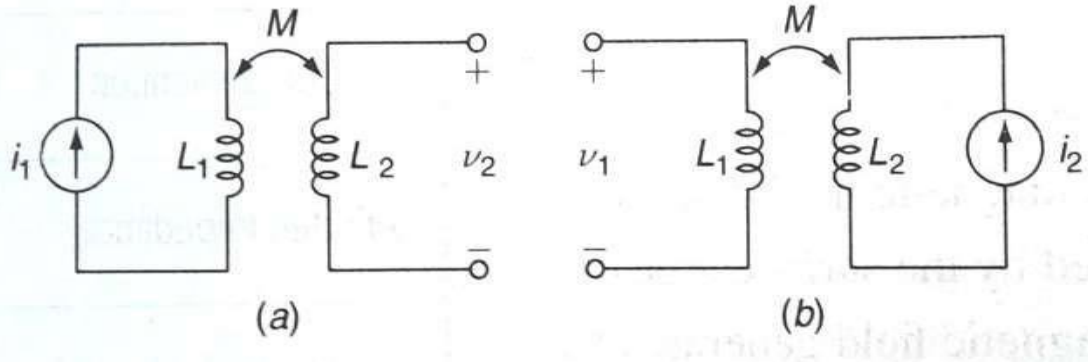


Figure 4.17 (a) A current i_1 through L_1 produces an open-circuit voltage v_2 across L_2 . (b) A current i_2 through L_2 produces an open-circuit voltage v_1 across L_1 .

The order of the subscripts on M_{21} indicates that a voltage response is produced at L_2 by a current source at L_1 . If the system is reversed, as indicated

in fig.(b) then we have

$$v_1(t) = M_{12} di_2(t)/dt \quad \text{-----}[2]$$

It can be proved that the two mutual inductances M_{12} and M_{21} are equal and thus, $M_{12} = M_{21} = M$. The existence of mutual coupling between two coils is indicated by a double-headed arrow, as shown in Fig. (a) and (b)

Mutual inductance is measured in Henrys and, like resistance, inductance, and capacitance, is a positive quantity. The voltage $M di/dt$, however, may appear as either a positive or a negative quantity depending on whether the current is increasing or decreasing at a particular instant of time.

Coefficient of coupling k : Is given by the relation $M = k\sqrt{L_1 L_2}$ and its value lies between 0 and

1. It can assume the maximum value of 1 when the two coils are wound on the same core such that flux produced by one coil completely links with the other coil. This is possible in well designed cores with high permeability. Transformers are designed to achieve a coefficient of coupling of 1.

Dot Convention:

The polarity of the voltage induced in a coil depends on the sense of winding of the coil. In the case of Mutual inductance it is indicated by use of a method called "**dot convention**". The dot

convention makes use of a large dot placed at one end of each of the two coils which are mutually coupled. Sign of the mutual voltage is determined as follows:

A current entering the dotted terminal of one coil produces an open circuit voltage with a positive voltage reference at the dotted terminal of the second coil.

Thus in Fig(a) i_1 enters the dotted terminal of L_1 , v_2 is sensed positively at the dotted terminal of L_2 , and $v_2 = M \frac{di_1}{dt}$.

It may not be always possible to select voltages or currents throughout a circuit so that the passive sign convention is everywhere satisfied; the same situation arises with mutual coupling. For example, it may be more convenient to represent v_2 by a positive voltage reference at the undotted terminal, as shown in Fig (b). Then $v_2 = -M \frac{di_1}{dt}$. Currents also may not always enter the dotted terminal as indicated by Fig (c) and (d). Then we note that:

A current entering the undotted terminal of one coil provides a voltage that is positively sensed at the undotted terminal of the second coil.

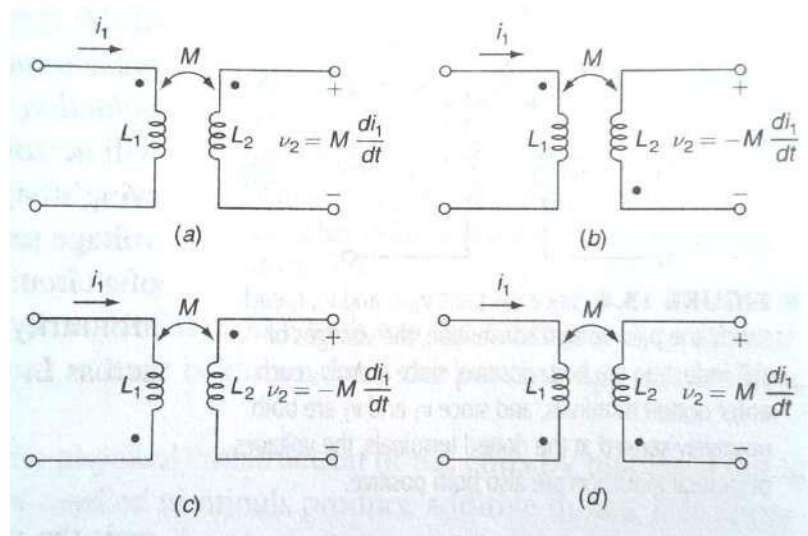


Figure 4.18 : (a) and (b) Current entering the dotted terminal of one coil produces a voltage that is sensed positively at the dotted terminal of the second coil. (c) and (d) Current entering the undotted terminal of one coil produces a voltage that is sensed positively at the undotted terminal of the second coil.

Important Concepts and formulae:

Resonance and Series RLC circuit:

$$\omega_r^2 = \omega_1 \omega_2 = 1/LC \therefore \omega_r = \sqrt{\omega_1 \omega_2} = 1/\sqrt{LC} \text{ BW} = R/2\pi L$$

$$Q = \omega_r L / R = 1 / \omega_r RC \quad \text{and in terms of } R, L \text{ and } C = (1/R) (\sqrt{L/C})$$

$$Q = f_r / BW \quad \text{i.e. inversely proportional to the BW}$$

$$\text{Voltage magnification Magnification} = Q = V_L / V \quad \text{or } V_C / V$$

Important points In Series RLC circuit at resonant frequency:

- The impedance of the circuit becomes purely resistive and minimum i.e $Z = R$
- The current in the circuit becomes maximum
- The magnitudes of the capacitive Reactance and Inductive Reactance become equal
- The voltage across the Capacitor becomes equal to the voltage across the Inductor at resonance and is Q times higher than the voltage across the resistor

Resonance and Parallel RLC circuit:

$$\omega_r^2 = \omega_1 \omega_2 = 1/LC \quad \therefore \omega_r = \sqrt{\omega_1 \omega_2} = 1/\sqrt{LC} \quad \text{same as in series RLC circuit}$$

$$BW = 1/2\pi RC$$

$$Q = R / \omega_r L = \omega_r RC \quad \text{and in terms of } R, L \text{ and } C = R (\sqrt{C/L}) \quad [\text{Inverse of what we got in Series RLC circuit}]$$

$$Q = f_r / BW \quad \text{In Parallel RLC also inversely proportional to the BW}$$

$$\text{Current Magnification} = Q = I_L / I \quad \text{or } I_C / I$$

Important points In Parallel RLC circuit at resonant frequency :

- The impedance of the circuit becomes resistive and maximum i.e $Z = R$
- The current in the circuit becomes minimum
- The magnitudes of the capacitive Reactance and Inductive Reactance become equal
- The current through the Capacitor becomes equal and opposite to the current through the Inductor at resonance and is Q times higher than the current through the resistor

Magnetic circuits :

Ampere's Law: $\mathbf{H} \cdot d\mathbf{l} = I_{net}$ and in the case of a simple closed magnetic path of a ferromagnetic material it simplifies to $\mathbf{Hl} = \mathbf{Ni}$ or $\mathbf{H} = \mathbf{Ni/l}$

Magnetic flux density:

$$\mathbf{B} = \mu \mathbf{H}$$

Magnetic field intensity:

$$\mathbf{H} = \mathbf{Ni/l}$$

Total magnetic flux intensity:

$$\mathbf{\Phi} = \mathbf{BA} = \mu \mathbf{HA} = \mu \mathbf{Ni A / l}$$

Reluctance of the magnetic circuit:

$$R = \text{mmf/Flux} = Ni / \Phi = l / \mu A$$

Faradays law of electromagnetic Induction:

Self induced e.m.f of a coil of N turns is given by:

$e_{\text{ind}} = -N d\Phi/dt = -L di/dt$ where L is the inductance of the coil of N turns with radius r and given by $L = (\mu N^2 \pi r^2 / l) i$

Equivalent Reluctance of a series Magnetic circuit:

$$R_{\text{eqseries}} = R_1 + R_2 + R_3 + \dots$$

Equivalent Reluctance of a Parallel Magnetic circuit:

$$1/R_{\text{eqparallel}} = 1/R_1 + 1/R_2 + 1/R_3 + \dots$$

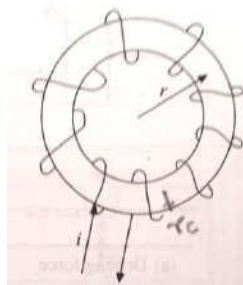
Coefficient of coupling k is given by the relation:

$$M = k \sqrt{L_1 L_2}$$

Illustrative examples:

Example 1: A toroidal core of radius 6 cms is having 1000 turns on it. The radius of cross section of the core 1cm. Find the current required to establish a total magnetic flux of 0.4mWb. When

- The core is nonmagnetic
- The core is made of iron having a relative permeability of 4000



Solution:

This problem can be solved by the direct application of the following formulae we know in magnetic circuits: $B = \Phi/A = \mu H$ and $H = Ni/l$

Where

B = magnetic flux density (Wb/mtr²)
(Wb))

Φ = Total magnetic flux

A = Cross sectional area of the core(mtr²)
(Henrys/mtr)

μ = $\mu_r \mu_0$ = Permeability

μ_r = Relative permeability of the material (Dimensionless)

μ_0 = free space permeability = $4\pi \times 10^{-7}$ Henrys/mtr

H = Magnetic field intensity AT/mtr

N = Number of turns of the coil

i = Current in the coil (Amps)
(mtrs)

l = Length of the coil

from the above relations we can get i as

$$i = H l / N = (1/\mu)(\Phi/A) l / N = (1/\mu)(\Phi/N) l / A = (1/\mu)(\Phi/N) * 2\pi r_T / \pi r_c^2 = [2r_T \Phi / \mu N r_c^2]$$

Where r_T is the radius of the toroid and r_c is the radius of cross section of the coil

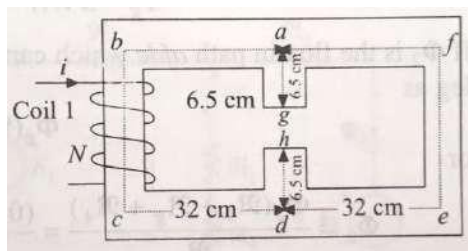
Now we can calculate the currents in the two cases by substituting the respective values. (a) Here $\mu =$

μ_0 . Therefore $i = (2 \times 6 \times 10^{-2} \times 4 \times 10^{-4}) / (4\pi \times 10^{-7} \times 1000 \times 10^{-4}) = 380$ Amps

(b) Here $\mu = \mu_r \mu_0$. Therefore $i = (2 \times 6 \times 10^{-2} \times 4 \times 10^{-4}) / (4000 \times 4\pi \times 10^{-7} \times 1000 \times 10^{-4}) = 0.095$ Amps

Ex.2: (a) Draw the electrical equivalent circuit of the magnetic circuit shown in the figure below. The area of the core is $2 \times 2 \text{ cm}^2$. The length of the air gap is 1 cm and lengths of the other limbs are shown in the figure. The relative permeability of the core is 4000.

(b) Find the value of the current ' i ' in the above example which produces a flux density of 1.2 Tesla in the air gap. The number of turns of the coil are 5000.



Solution: (a)

To draw the equivalent circuit we have to find the Reluctances of the various flux paths independently.

The reluctance of the path $abcd$ is given by:

$$R_1 = \text{length of the path } abcd / \mu_r \mu_0 A$$

$$= (32 \times 10^{-2}) / (4\pi \times 10^{-7} \times 4000 \times 4 \times 10^{-4})$$

$$= 1.59 \times 10^5 \text{ AT/Wb}$$

The reluctance of the path $afed$ is equal to the reluctance of the path $abcd$ since it has the same length, same permeability and same cross sectional area. Thus $R_1 = R_2$

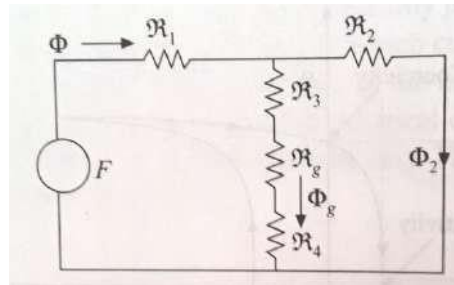
Similarly the reluctance of the path ag (R_3) is equal to that of the path hd (R_4) and can be calculated as:

$$R_3 = R_4 = (6.5 \times 10^{-2}) / (4\pi \times 10^{-7} \times 4000 \times 4 \times 10^{-4}) = 0.32 \times 10^5 \text{ AT/Wb}$$

The reluctance of the air gap path gh R_G can be calculated as : $R_G = \text{length of the air gap path} / \mu_0 A$

(Here it is to be noted that μ is to be taken as μ_0 only and μ_r should not be included) $R_G = (1 \times 10^{-2}) / (4\pi \times 10^{-7} \times 4 \times 10^{-4}) = 198.94 \times 10^5 \text{ AT/Wb}$

The equivalent electrical circuit is shown in the figure below with the values of the reluctances as given below the circuit diagram.



$$R_1 = R_2 = 1.59 \times 10^5 \text{ AT/Wb}$$

$$R_3 = R_4 = 0.32 \times 10^5 \text{ AT/Wb}$$

$$R_G = 198.94 \times 10^5$$

Solution: (b)

This problem is solved in the following steps:

- First the flux through the air gap Φ_G is found out.** The flux in the air gap Φ_G is given by the product of the Flux density in the air gap B and the cross sectional area of the core in that region A . Hence $\Phi_G = B.A = 1.2 \times 4 \times 10^{-4} = 0.00048 \text{ Wb}$

It is to be noted here that the same flux would be passing through the reluctances

R_3, R_G &

R_4

- Next, the Flux in the path $afed$ Φ_2 is to be found out.** This can be found out by noticing that the mmf across the reluctance R_2 is same as the mmf across the sum of the reluctances R_3, R_G , and R_4 coming in parallel with R_4 . Hence by equating them we get

$$\Phi_G (R_3 + R_G + R_4) = \Phi_2 R_2 \text{ from which we get } \Phi_2 = \Phi_G (R_3 + R_G + R_4) / R_2$$

$$\text{Hence } \Phi_2 = [0.00048 \times (0.32 + 198.94 + 0.32) \times 10^5] / 1.59 \times 10^5 = 0.06025 \text{ Wb}$$

1. **Next , the total flux Φ flowing through the reluctance of the path abcd R_1 produced by the winding is to be found out.** This is the sum of the air gap flux Φ_G and the flux in the outer limb of the core Φ_2 :
i.e $\Phi = \Phi_G + \Phi_2 = (0.00048 + 0.06025) = 0.0607 \text{ Wb}$
2. **Next , The total mmf F given by $F = Ni$ is to be found out .** This is also equal to the sum of the mmfs across the reluctances R_1 and R_2 [or $(R_3 + R_G + R_4)$] = $\Phi R_1 + \Phi_2 R_2$ from which we can get 'i' as: 'i' = $(\Phi R_1 + \Phi_2 R_2) / N = [0.0607 \times 1.59 \times 10^5 + 0.06025 \times 1.59 \times 10^5] / 5000 = 3.847 \text{ Amps}$

$$i = 3.847 \text{ Amps}$$

UNIT-IV TRANSMISSION LINES-I

- Types of transmission lines
- Transmission line Parameters- Primary & Secondary Constants
- Transmission Line Equations
- Expressions for Characteristics Impedance
- Propagation Constant
- Phase and Group Velocities
- Infinite Line Concepts
- Lossless transmission line
- Distortion
- Condition for Distortionless transmission
- Minimum Attenuation
- Illustrative Problems.

TRANSMISSION LINE THEORY

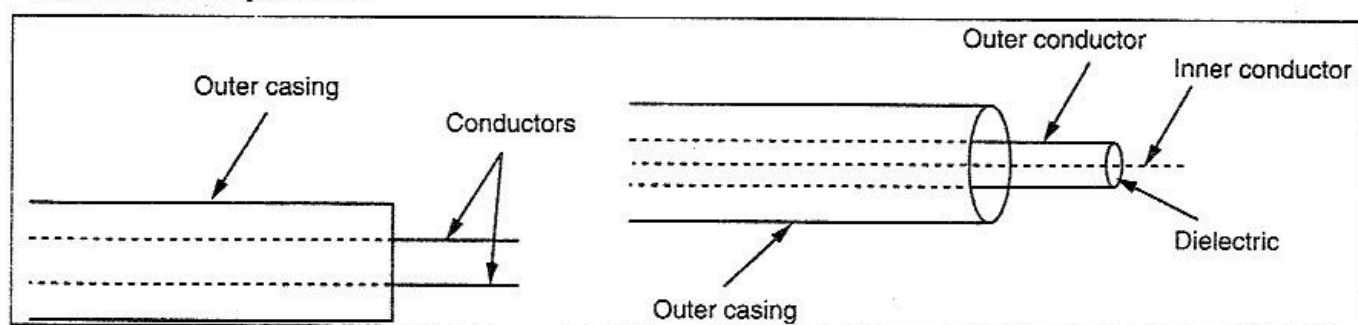
1.1. INTRODUCTION

The transfer of energy from one point to another takes place through either wave guides or transmission lines. Transmission lines always consist of atleast two separate conductors between which a voltage can exist, but the wave guides involve only one conductor; for example, a hollow rectangular or circular waveguide within which the wave propagates. Transmission lines are a means of conveying power from one point to another. There are two types of commonly used transmission lines.

1. Parallel wire (balanced) line
2. Coaxial (unbalanced) line

Parallel wire line : It is a common form of transmission line known as open wire line as shown in Fig. 1.1(a). It is employed where balanced properties are required. Telephone lines, line connecting between folded dipole antenna and TV receiver are good examples of parallel or balanced or open wire line. The parallel wire lines are not used for microwave transmission.

Coaxial line : Coaxial lines consist of inner and outer conductor spacers of dielectric as shown in Fig. 1.1(b). It is used when unbalanced properties are needed, as in the interconnection of a broadcast transmitter to its grounded antenna. It is employed at UHF and microwave frequencies.



(a) Parallel wire (balanced) line

(b) Coaxial (unbalanced) line

Fig. 1.1. Transmission lines

1.2. TRANSMISSION LINE AS CASCADED T SECTIONS

To study the behaviour of transmission line, a transmission can be considered to be made up of a number of identical symmetrical T sections connected in series as in Fig.1.2. If the last section is terminated with its characteristic impedance, the input impedance at the first section is Z_0 . Each section is terminated by the input impedance of the following section.

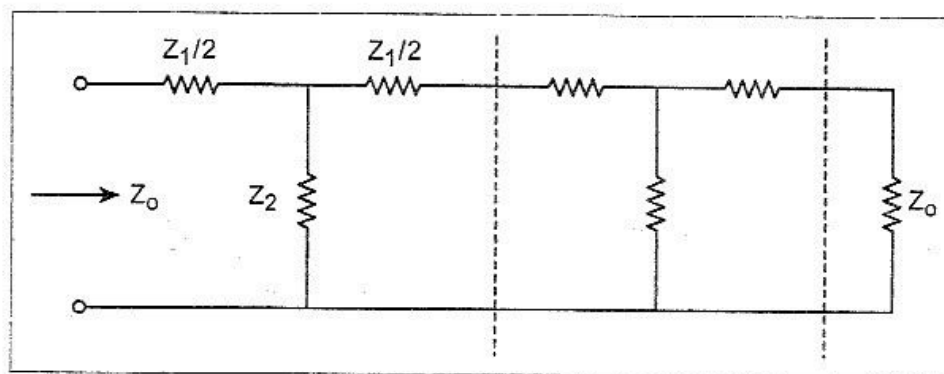


Fig. 1.2. A line of cascaded T sections

The characteristic impedance for a T section is

$$Z_{0T} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4 Z_2} \right)}$$

If 'n' number of T sections are cascaded and if the sending and receiving currents are I_S and I_R respectively, then

$$I_S = I_R e^{n\gamma}$$

where γ is the propagation constant for one T section.

$$\gamma = \alpha + j\beta$$

$$e^\gamma = e^{\alpha + j\beta} = 1 + \frac{Z_1}{2 Z_2} + \sqrt{\frac{Z_1}{Z_2} \left(1 + \frac{Z_1}{4 Z_2} \right)}$$

One T section representing an incremental length Δx of the line has a series impedance $Z_1 = Z \Delta x$ and shunt impedance $Z_2 = \frac{1}{Y \Delta x}$. The characteristic impedance of any small T section is that of the line as a whole.

$$Z_0 = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4 Z_2} \right)}$$

Substituting the values of Z_1 and Z_2 ,

$$\begin{aligned} Z_0 &= \sqrt{\frac{Z \Delta x}{Y \Delta x} \left(1 + \frac{Z \Delta x Y \Delta x}{4} \right)} \\ &= \sqrt{\frac{Z}{Y} \left(1 + \frac{ZY (\Delta x)^2}{4} \right)} \end{aligned}$$

If Δx tends to zero, then Z_0 becomes,

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$\sqrt{\frac{Z_1}{Z_2} \left(1 + \frac{Z_1}{4Z_2} \right)} = \sqrt{\frac{Z_1}{Z_2} \left(1 + \frac{Z_1}{4Z_2} \right)^{\frac{1}{2}}}$$

By the binomial theorem,

$$\sqrt{\frac{Z_1}{Z_2} \left(1 + \frac{Z_1}{4Z_2} \right)} = \sqrt{\frac{Z_1}{Z_2} \left[1 + \frac{1}{2} \left(\frac{Z_1}{4Z_2} \right) - \frac{1}{8} \left(\frac{Z_1}{4Z_2} \right)^2 + \dots \right]}$$

Substituting this value in e^γ equation,

$$\begin{aligned} e^\gamma &= 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} \left(1 + \frac{Z_1}{4Z_2} \right)} \\ &= 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2}} + \frac{1}{8} \left(\frac{Z_1}{Z_2} \right) \sqrt{\frac{Z_1}{Z_2}} - \frac{1}{128} \left(\frac{Z_1}{Z_2} \right)^2 \sqrt{\frac{Z_1}{Z_2}} + \dots \\ &= 1 + \sqrt{\frac{Z_1}{Z_2}} + \frac{1}{2} \left(\sqrt{\frac{Z_1}{Z_2}} \right)^2 + \frac{1}{8} \left(\sqrt{\frac{Z_1}{Z_2}} \right)^3 - \frac{1}{128} \left(\sqrt{\frac{Z_1}{Z_2}} \right)^5 + \dots \end{aligned}$$

When applied to the incremental length of line Δx , then $Z_1 = Z \Delta x$, $Z_2 = \frac{1}{Y \Delta x}$ and propagation constant becomes $\gamma \Delta x$,

$$e^{\gamma \Delta x} = 1 + \sqrt{ZY} \Delta x + \frac{1}{2} (\sqrt{ZY})^2 (\Delta x)^2 + \frac{1}{8} (\sqrt{ZY})^3 (\Delta x)^3 - \frac{1}{128} (\sqrt{ZY})^5 (\Delta x)^5$$

Series expansion for an exponential $e^{\gamma \Delta x}$ is

$$e^{\gamma \Delta x} = 1 + \gamma \Delta x + \frac{\gamma^2 (\Delta x)^2}{2!} + \frac{\gamma^3 (\Delta x)^3}{3!} + \dots$$

Equating the above two expressions,

$$\sqrt{ZY} \Delta x + \frac{(\sqrt{ZY})^2 (\Delta x)^2}{2} + \frac{(\sqrt{ZY})^3 (\Delta x)^3}{8} + \dots = \gamma \Delta x + \frac{\gamma^2 (\Delta x)^2}{2} + \frac{\gamma^3 (\Delta x)^3}{6} + \dots$$

$$\gamma + \frac{\gamma^2 \Delta x}{2} + \frac{\gamma^3 (\Delta x)^2}{6} + \dots = \sqrt{ZY} + \frac{(\sqrt{ZY})^2}{2} \Delta x + \frac{(\sqrt{ZY})^3 (\Delta x)^2}{8} + \dots$$

If Δx tends to zero then,

$$\gamma = \sqrt{ZY}$$

This is the value of propagation constant in terms of Z and Y .

Since each conductor of transmission line has a certain length and diameter, it must have resistance and inductance; moreover the two conductors are separated by a dielectric medium (say, air), therefore there must be a capacitance between them. This dielectric between the conducting wires may not be perfect, and hence a leakage current will flow creating leakage (shunt) capacitance between the conductors. These four parameters resistance (R), inductance (L), capacitance (C) and conductance (G), all distributed along the lines are known as

distributed parameters. The equivalent circuit diagram of transmission line is shown in Fig. 1.3.

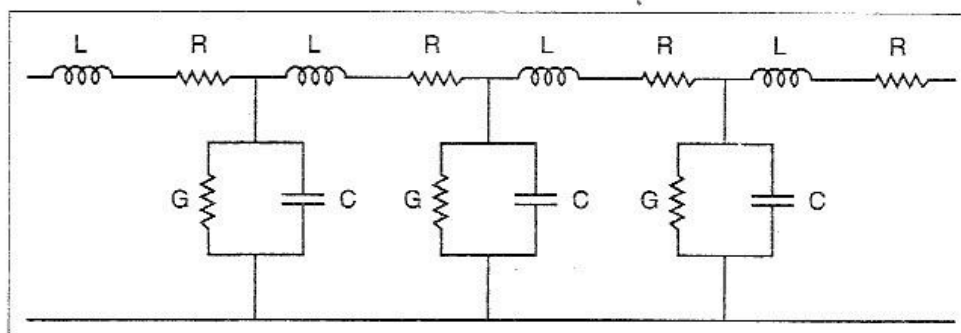


Fig. 1.3. Equivalent circuit diagram of transmission line

The four line parameters resistance (R), inductance (L), capacitance (C) and conductance (G) are also known as *primary constants* of the transmission line.

Resistance (R) is defined as the loop resistance per unit length of the transmission line. It is measured in ohms/km.

Inductance (L) is defined as the loop inductance per unit length of the transmission line. It is measured in Henries/km.

Capacitance (C) is defined as the shunt capacitance per unit length between the two transmission lines. It is measured in Farads/km.

Conductance (G) is defined as the shunt conductance per unit length between the two transmission lines. It is measured in mhos/km.

1.3. TRANSMISSION LINE EQUATION

Transmission line is a conductive method of guiding electrical energy from one place to another. A uniform transmission line can be considered to be made up of an infinite number of T sections, each of infinitesimal size dx . The equivalent circuit of T section of transmission line is shown in Fig. 1.4.

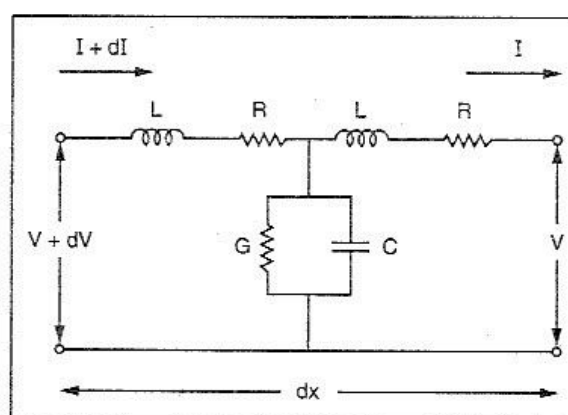


Fig. 1.4. Equivalent circuit of T section of Transmission line

The parameters R, L, G and C are distributed throughout the transmission line. The constants of an incremental length dx of a line are shown in Fig. 1.4. The series impedance per unit length and shunt admittance per unit length are given by

$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

Consider a T section of transmission line of length dx . Let $V + dV$ be the voltage and $I + dI$ be the current at one end of T section. Let V be the voltage and I be the current at the other end of this section.

The series impedance of a small section dx is $(R + j\omega L) dx$. The shunt admittance of this section dx is $(G + j\omega C) dx$.

The voltage drop across the series impedance of T sections *i.e.*, the potential difference between the two ends of T section is

$$\begin{aligned} V + dV - V &= I(R + j\omega L) dx \\ dV &= I(R + j\omega L) dx \\ \frac{dV}{dx} &= I(R + j\omega L) \quad \dots (1.1) \\ \frac{dV}{dx} &= IZ \end{aligned}$$

The current difference between the two ends of T section is due to the voltage drop across the shunt admittance.

$$\begin{aligned} I + dI - I &= V(G + j\omega C) dx \\ dI &= V(G + j\omega C) dx \\ \frac{dI}{dx} &= V(G + j\omega C) \quad \dots (1.2) \\ \frac{dI}{dx} &= VY \end{aligned}$$

Differentiating equation (1.1) w.r.t. 'x',

$$\frac{d^2V}{dx^2} = (R + j\omega L) \frac{dI}{dx}$$

Substituting the value of $\frac{dI}{dx}$ in the above equation

$$\frac{d^2V}{dx^2} = (R + j\omega L)(G + j\omega C) V \quad \dots (1.3)$$

Differentiating equation (1.2) w.r.t. 'x'

$$\frac{d^2I}{dx^2} = (G + j\omega C) \frac{dV}{dx}$$

Substituting the value of $\frac{dV}{dx}$ in the above equation

$$\frac{d^2I}{dx^2} = (R + j\omega L)(G + j\omega C) I \quad \dots (1.4)$$

But propagation constant is given by

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{ZY} \quad 113$$

Substituting the value of γ in equation (1.3) and (1.4),

$$\text{then } \frac{d^2V}{dx^2} = \gamma^2 V$$

$$\frac{d^2I}{dx^2} = \gamma^2 I$$

The solutions of the above linear differential equations are

$$V = A e^{\gamma x} + B e^{-\gamma x} \quad \dots (1.5)$$

$$I = C e^{\gamma x} + D e^{-\gamma x} \quad \dots (1.6)$$

where A, B, C and D are arbitrary constants.

Differentiating the equation (1.5), w.r.t. 'x'

$$\frac{dV}{dx} = A \gamma e^{\gamma x} - B \gamma e^{-\gamma x}$$

$$\text{But } \frac{dV}{dx} = IZ$$

$$\begin{aligned} IZ &= A \gamma e^{\gamma x} - B \gamma e^{-\gamma x} \\ &= A \sqrt{ZY} e^{\sqrt{ZY} x} - B \sqrt{ZY} e^{-\sqrt{ZY} x} \quad [\because \gamma = \sqrt{ZY}] \end{aligned}$$

$$I = A \sqrt{\frac{Y}{Z}} e^{\sqrt{ZY} x} - B \sqrt{\frac{Y}{Z}} e^{-\sqrt{ZY} x} \quad \dots (1.7)$$

Similarly, differentiating the equation (1.6) w.r.t. 'x'

$$\frac{dI}{dx} = C \gamma e^{\gamma x} - D \gamma e^{-\gamma x}$$

$$\text{But } \frac{dI}{dx} = VY$$

$$\begin{aligned} VY &= C \gamma e^{\gamma x} - D \gamma e^{-\gamma x} \\ &= C \sqrt{ZY} e^{\sqrt{ZY} x} - D \sqrt{ZY} e^{-\sqrt{ZY} x} \\ V &= C \sqrt{\frac{Z}{Y}} e^{\sqrt{ZY} x} - D \sqrt{\frac{Z}{Y}} e^{-\sqrt{ZY} x} \quad \dots (1.8) \end{aligned}$$

Since the distance x is measured from the receiving end of the transmission line,

$$x = 0, \quad \therefore I = I_R$$

$$V = V_R$$

$$V_R = I_R Z_R$$

where I_R is the current in the receiving end of line

V_R is the voltage across the receiving end of the lines

Z_R is the impedance of receiving end

Substituting this condition in equations (1.5), (1.6), (1.7) and (1.8).

$$V_R = A + B \quad \dots (1.9)$$

$$I_R = C + D \quad \dots (1.10)$$

$$I_R = A \sqrt{\frac{Y}{Z}} - B \sqrt{\frac{Y}{Z}} \quad \dots (1.11)$$

$$V_R = C \sqrt{\frac{Z}{Y}} - D \sqrt{\frac{Z}{Y}} \quad \dots (1.12)$$

To solve these equations,

$$\text{Let } x = \sqrt{\frac{Z}{Y}} \quad \text{and} \quad \frac{1}{x} = \sqrt{\frac{Y}{Z}}$$

$$\begin{aligned} \text{Then } I_R &= \frac{A}{x} - \frac{B}{x} \\ &= \frac{1}{x} (A - B) \end{aligned}$$

$$\text{But } I_R = C + D$$

$$C + D = \frac{1}{x} (A - B)$$

$$Cx + Dx = A - B$$

$$A - B = Cx + Dx \quad \dots (1.13)$$

Similarly, equation (1.12) becomes,

$$V_R = Cx - Dx$$

$$\text{But } V_R = A + B$$

$$A + B = Cx - Dx \quad \dots (1.14)$$

$$A - B = Cx + Dx \quad \dots (1.13)$$

Adding the equations (1.13) and (1.14),

$$2A = 2Cx$$

$$A = Cx$$

Similarly subtracting the equations (1.13) and (1.14),

$$2B = -2x Dx$$

$$B = -Dx$$

Substituting the values of A and B in the following equations.

$$V_R = A + B$$

$$= Cx - Dx$$

$$\text{But } I_R = C + D$$

$$I_R x = Cx + Dx \quad \dots (1.15)$$

$$V_R = Cx - Dx \quad \dots (1.16)$$

Adding the equations (1.15) and (1.16),

$$2Cx = I_R x + V_R$$

$$C = \frac{I_R}{2} + \frac{V_R}{2x}$$

$$\therefore C = \frac{I_R}{2} + \frac{V_R}{2} \sqrt{\frac{Y}{Z}} \quad \dots (1.17) \quad [\because x = \sqrt{\frac{Z}{Y}}]$$

Subtracting the equations (1.15) and (1.16),

$$2Dx = I_R x - V_R$$

$$D = \frac{I_R}{2} - \frac{V_R}{2x}$$

$$\therefore D = \frac{I_R}{2} - \frac{V_R}{2} \sqrt{\frac{Y}{Z}} \quad \dots (1.18)$$

$$\text{But } A = Cx$$

$$A = \frac{I_R}{2} x + \frac{V_R}{2}$$

$$\therefore A = \frac{V_R}{2} + \frac{I_R}{2} \sqrt{\frac{Z}{Y}} \quad \dots (1.19)$$

$$B = -Dx$$

$$B = -\frac{I_R}{2} x + \frac{V_R}{2}$$

$$\therefore B = \frac{V_R}{2} - \frac{I_R}{2} \sqrt{\frac{Z}{Y}} \quad \dots (1.20)$$

The characteristic impedance is defined as

$$\begin{aligned} Z_0 &= \sqrt{\frac{Z}{Y}} \\ &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \end{aligned} \quad \dots (1.21)$$

Substituting the value of Z_0 in equations (1.19), (1.20), (1.17) and (1.18),

$$A = \frac{V_R}{2} + \frac{I_R}{2} \sqrt{\frac{Z}{Y}}$$

$$A = \frac{V_R}{2} + \frac{V_R}{2 Z_R} Z_0$$

$$A = \frac{V_R}{2} \left[1 + \frac{Z_0}{Z_R} \right] \quad \dots (1.22)$$

$$B = \frac{V_R}{2} - \frac{I_R}{2} \sqrt{\frac{Z}{Y}}$$

$$= \frac{V_R}{2} - \frac{V_R}{2 Z_R} Z_0$$

$$B = \frac{V_R}{2} \left[1 - \frac{Z_0}{Z_R} \right] \quad \dots (1.23)$$

$$C = \frac{I_R}{2} + \frac{V_R}{2} \sqrt{\frac{Y}{Z}}$$

$$= \frac{I_R}{2} + \frac{I_R Z_R}{2 Z_0} \quad [\because V_R = I_R Z_R]$$

$$C = \frac{I_R}{2} \left[1 + \frac{Z_R}{Z_0} \right] \quad \dots (1.24)$$

$$D = \frac{I_R}{2} - \frac{V_R}{2} \sqrt{\frac{Y}{Z}}$$

$$= \frac{I_R}{2} - \frac{I_R Z_R}{2 Z_0}$$

$$D = \frac{I_R}{2} \left[1 + \frac{Z_R}{Z_0} \right] \quad \dots (1.25)$$

Substituting the values of A, B, C and D in equations (1.5) and (1.6), the solutions of the differential equations are

$$V = \frac{V_R}{2} \left(1 + \frac{Z_0}{Z_R} \right) e^{\sqrt{ZY}x} + \frac{V_R}{2} \left(1 - \frac{Z_0}{Z_R} \right) e^{-\sqrt{ZY}x} \quad \dots (1.26)$$

$$I = \frac{I_R}{2} \left(1 + \frac{Z_R}{Z_0} \right) e^{\sqrt{ZY}x} + \frac{I_R}{2} \left(1 - \frac{Z_R}{Z_0} \right) e^{-\sqrt{ZY}x} \quad \dots (1.27)$$

$$V = \frac{V_R}{2} \left[\left(1 + \frac{Z_0}{Z_R} \right) e^{\sqrt{ZY}x} + \left(1 - \frac{Z_0}{Z_R} \right) e^{-\sqrt{ZY}x} \right] \quad \dots (1.28)$$

$$I = \frac{I_R}{2} \left[\left(1 + \frac{Z_R}{Z_0} \right) e^{\sqrt{ZY}x} + \left(1 - \frac{Z_R}{Z_0} \right) e^{-\sqrt{ZY}x} \right] \quad \dots (1.29)$$

After simplification,

$$V = \frac{V_R}{2} e^{\sqrt{ZY}x} + \frac{V_R}{2} \frac{Z_0}{Z_R} e^{\sqrt{ZY}x} + \frac{V_R}{2} e^{-\sqrt{ZY}x} - \frac{V_R}{2} \frac{Z_0}{Z_R} e^{-\sqrt{ZY}x}$$

$$I = \frac{I_R}{2} e^{\sqrt{ZY}x} + \frac{I_R}{2} \frac{Z_R}{Z_0} e^{\sqrt{ZY}x} + \frac{I_R}{2} e^{-\sqrt{ZY}x} - \frac{I_R}{2} \frac{Z_R}{Z_0} e^{-\sqrt{ZY}x}$$

$$V = V_R \left(\frac{e^{\sqrt{ZY}x} + e^{-\sqrt{ZY}x}}{2} \right) + I_R Z_0 \left(\frac{e^{\sqrt{ZY}x} - e^{-\sqrt{ZY}x}}{2} \right) \quad [\because V_R = I_R Z_R]$$

$$I = I_R \left(\frac{e^{\sqrt{ZY}x} + e^{-\sqrt{ZY}x}}{2} \right) + \frac{V_R}{Z_0} (e^{\sqrt{ZY}x} - e^{-\sqrt{ZY}x}) \quad \left[\because I_R = \frac{V_R}{Z_R} \right]$$

Then equations can be written in terms of hyperbolic functions.

$$V = V_R \cosh \sqrt{ZY} x + I_R Z_0 \sinh \sqrt{ZY} x \quad \dots (1.30)$$

$$I = I_R \cosh \sqrt{ZY} x + \frac{V_R}{Z_0} \sinh \sqrt{ZY} x \quad \dots (1.31)$$

These are the equations for voltage and current of a transmission line at any distance 'x' from the receiving end of transmission line.

The equations for voltage and current at the sending end of a transmission line of length 'l' are given by

$$V_S = V_R \cosh \sqrt{ZY} l + \frac{V_R}{Z_R} Z_0 \sinh \sqrt{ZY} l \quad \left[\because I_R = \frac{V_R}{Z_R} \right]$$

$$I_S = I_R \cosh \sqrt{ZY} l + \frac{I_R Z_R}{Z_0} \sinh \sqrt{ZY} l \quad [\because V_R = I_R Z_R]$$

$$V_S = V_R \left[\cosh \sqrt{ZY} l + \frac{Z_0}{Z_R} \sinh \sqrt{ZY} l \right] \quad \dots (1.32)$$

$$I_S = I_R \left[\cosh \sqrt{ZY} l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY} l \right] \quad \dots (1.33)$$

1.4. WAVELENGTH AND VELOCITY OF PROPAGATION

The propagation constant (γ) and characteristic impedance (Z_0) are called secondary constants of a transmission line.

Propagation constant is usually a complex quantity.

$$\gamma = \alpha + j\beta$$

where α is the attenuation constant.

β is the phase shift.

$$\gamma = \sqrt{ZY}$$

where $Z = R + j\omega L$

$$Y = G + j\omega C$$

The characteristic impedance of the transmission line is also a complex quantity.

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \dots (1.34)$$

Propagation constant is $\gamma = \alpha + j\beta$

$$= \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\alpha + j\beta = \sqrt{RG - \omega^2 LC + j\omega(LG + RC)} \quad \dots (1.35)$$

Squaring on both sides,

$$(\alpha + j\beta)^2 = RG - \omega^2 LC + j\omega(LG + RC)$$

$$\alpha^2 - \beta^2 + 2j\alpha\beta = RG - \omega^2 LC + j\omega(LG + RC) \quad \dots (1.36)$$

Equating real parts,

$$\alpha^2 - \beta^2 = RG - \omega^2 LC$$

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \quad \dots (1.37)$$

Equating imaginary parts,

$$2\alpha\beta = \omega(LG + RC)$$

Squaring on both sides,

$$4\alpha^2\beta^2 = \omega^2(LG + RC)^2$$

$$\alpha^2\beta^2 = \frac{\omega^2}{4}(LG + RC)^2$$

Substituting the value of α^2 [eqn. (1.37)] in the above equation,

$$(\beta^2 + RG - \omega^2 LC)\beta^2 = \frac{\omega^2}{4}(LG + RC)^2$$

$$\beta^4 + \beta^2(RG - \omega^2 LC) - \frac{\omega^2}{4}(LG + RC)^2 = 0$$

The solution of the quadratic equation is

$$\beta^2 = \frac{-(RG - \omega^2 LC) \pm \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}$$

By neglecting the negative values,

$$\therefore \beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}} \quad \dots (1.38)$$

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \quad \dots (1.37)$$

Substituting the value of β [eqn. (1.38)] in the above equation,

$$\begin{aligned} \alpha^2 &= \frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2} + RG - \omega^2 LC \\ &= \frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2} \\ \therefore \alpha &= \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}} \quad \dots (1.39) \end{aligned}$$

For a perfect transmission line $R = 0$ and $G = 0$,

$$\beta^2 = \omega^2 LC$$

$$\therefore \beta = \omega \sqrt{LC} \quad [\text{only positive value}]$$

Velocity :

The velocity of propagation is given by,

$$v = \lambda f$$

$$= 2\pi f \frac{\lambda}{2\pi}$$

$$v = \frac{\omega}{\beta}$$

$$[\because \beta = \frac{2\pi}{\lambda} \text{ and } \omega = 2\pi f]$$

Substituting the value of $\beta = \omega \sqrt{LC}$

$$\therefore v = \frac{\omega}{\omega \sqrt{LC}}$$

$$v = \frac{1}{\sqrt{LC}}$$

This is the velocity of propagation for an ideal line.

Wavelength :

The distance travelled by the wave along the line while the phase angle is changing through 2π radians is called wavelength.

$$\beta\lambda = 2\pi$$

$$\lambda = \frac{2\pi}{\beta} \quad \text{or} \quad \lambda = \frac{v}{f}$$

1.5. INPUT IMPEDANCE AND TRANSFER IMPEDANCE OF TRANSMISSION LINE

Input impedance :

The equations for voltage and current at the sending end of a transmission line of length 'l' are given by

$$V_S = V_R \left(\cosh \sqrt{ZY} l + \frac{Z_0}{Z_R} \sinh \sqrt{ZY} l \right) \quad \dots (1.32)$$

$$I_S = I_R \left(\cosh \sqrt{ZY} l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY} l \right) \quad \dots (1.33)$$

The input impedance of the transmission line is,

$$\begin{aligned} Z_S &= \frac{V_S}{I_S} \\ &= \frac{V_R \left(\cosh \sqrt{ZY} l + \frac{Z_0}{Z_R} \sinh \sqrt{ZY} l \right)}{I_R \left(\cosh \sqrt{ZY} l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY} l \right)} \\ &= \frac{I_R Z_R \left(\cosh \sqrt{ZY} l + \frac{Z_0}{Z_R} \sinh \sqrt{ZY} l \right)}{I_R \left(\cosh \sqrt{ZY} l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY} l \right)} \\ Z_S &= \frac{Z_0 (Z_R \cosh \sqrt{ZY} l + Z_0 \sinh \sqrt{ZY} l)}{(Z_0 \cosh \sqrt{ZY} l + Z_R \sinh \sqrt{ZY} l)} \quad \dots (1.40) \end{aligned}$$

$$\text{Let } \sqrt{ZY} = \gamma$$

The input impedance of the line is

$$Z_S = Z_0 \left[\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right]$$

$$\text{or } Z_S = Z_0 \left[\frac{Z_R + Z_0 \tanh \gamma l}{Z_0 + Z_R \tanh \gamma l} \right]$$

In a different form, the equations for voltage and current at transmitting end of a line is given by equations (1.28) and (1.29),

$$V_S = \frac{V_R}{2} \left[\left(1 + \frac{Z_0}{Z_R} \right) e^{\sqrt{ZY} l} + \left(1 - \frac{Z_0}{Z_R} \right) e^{-\sqrt{ZY} l} \right] \quad \dots (1.28)$$

$$I_S = \frac{I_R}{2} \left[\left(1 + \frac{Z_R}{Z_0} \right) e^{\sqrt{ZY} l} + \left(1 - \frac{Z_R}{Z_0} \right) e^{-\sqrt{ZY} l} \right] \quad \dots (1.29)$$

$$\begin{aligned} \text{or } V_S &= \frac{V_R}{2} \left[\left(\frac{Z_R + Z_0}{Z_R} \right) e^{\sqrt{ZY}l} + \left(\frac{Z_R - Z_0}{Z_R} \right) e^{-\sqrt{ZY}l} \right] \\ I_S &= \frac{I_R}{2} \left[\left(\frac{Z_R + Z_0}{Z_0} \right) e^{\sqrt{ZY}l} + \left(\frac{Z_0 - Z_R}{Z_0} \right) e^{-\sqrt{ZY}l} \right] \\ \text{or } V_S &= \left(\frac{V_R}{2} \right) \left(\frac{Z_R + Z_0}{Z_R} \right) \left[e^{\sqrt{ZY}l} + \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\sqrt{ZY}l} \right] \quad \dots (1.41) \end{aligned}$$

$$I_S = \frac{I_R}{2} \left(\frac{Z_R + Z_0}{Z_0} \right) \left[e^{\sqrt{ZY}l} - \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\sqrt{ZY}l} \right] \quad \dots (1.42)$$

The input impedance of the transmission line is given by,

$$Z_S = \frac{V_S}{I_S} = Z_0 \left[\frac{e^{\sqrt{ZY}l} + \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\sqrt{ZY}l}}{e^{\sqrt{ZY}l} - \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\sqrt{ZY}l}} \right] \quad [\because V_R = I_R Z_R] \quad \dots (1.43)$$

$$\text{Let } \sqrt{ZY} = \gamma$$

The input impedance of the transmission line is,

$$Z_S = Z_0 \left[\frac{e^{\gamma l} + \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\gamma l}}{e^{\gamma l} - \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\gamma l}} \right] \quad \dots (1.44)$$

If the line is terminated with its characteristic impedance *i.e.*, $Z_R = Z_0$, then the input impedance becomes equal to its characteristic impedance.

$$Z_S = Z_0$$

The input impedance of an infinite line is determined by letting $l \rightarrow \infty$.

$$\therefore Z_S = Z_0$$

It is found that a line of finite length, terminated with its characteristic impedance, appears to the transmitting end generator as an infinite line. A finite line terminated with Z_0 and an infinite line are same by measurements at the source.

$$\text{If } K = \frac{Z_R - Z_0}{Z_R + Z_0}, \text{ then}$$

$$Z_S = Z_0 \left[\frac{e^{\gamma l} + K e^{-\gamma l}}{e^{\gamma l} - K e^{-\gamma l}} \right] \quad \dots (1.45)$$

Transfer impedance :

Transfer impedance is used to determine the current at the receiving end if voltage at transmitting end is known. Transfer impedance of a transmission line is defined as the ratio of voltage at the sending end (transmitted voltage) to the current at the receiving end (received current).

$$Z_T = \frac{V_S}{I_R}$$

Equation (1.41) becomes

$$V_S = \frac{V_R (Z_R + Z_0)}{2 Z_R} (e^{\gamma l} + K e^{-\gamma l})$$

$$V_S = \frac{I_R (Z_R + Z_0)}{2} (e^{\gamma l} + K e^{-\gamma l}) \quad [\because V_R = I_R Z_R]$$

$$\begin{aligned} Z_T = \frac{V_S}{I_R} &= \frac{Z_R + Z_0}{2} (e^{\gamma l} + K e^{-\gamma l}) \\ &= \frac{Z_R + Z_0}{2} \left(e^{\gamma l} + \frac{Z_R - Z_0}{Z_R + Z_0} e^{-\gamma l} \right) \\ &= \left(\frac{Z_R + Z_0}{2} \right) e^{\gamma l} + \left(\frac{Z_R - Z_0}{2} \right) e^{-\gamma l} \\ &= Z_R \left(\frac{e^{\gamma l} + e^{-\gamma l}}{2} \right) + Z_0 \left(\frac{e^{\gamma l} - e^{-\gamma l}}{2} \right) \\ &= Z_R \cosh \gamma l + Z_0 \sinh \gamma l \\ &\quad \left[\because \frac{e^{\gamma l} + e^{-\gamma l}}{2} = \cosh \gamma l \text{ and } \frac{e^{\gamma l} - e^{-\gamma l}}{2} = \sinh \gamma l \right] \\ Z_T &= Z_R \cosh \gamma l + Z_0 \sinh \gamma l \end{aligned}$$

1.6. LINE DISTORTION

Signal (e.g., voice) transmitted over a transmission line is normally complex and consists of many frequency components. Such voice voltage will not have all frequencies transmitted with equal attenuation and equal time delay, the received waveform will not be identical with the input waveform at the sending end. This variation is known as **distortion**. There are two types of line distortions. They are frequency distortion and delay distortion.

Frequency Distortion : A complex (voice) voltage transmitted on a transmission line will not be attenuated equally and the received waveform will not be identical with the input waveform at the transmitting end. This variation is known as frequency distortion.

The attenuation constant is given by

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

α is a function of frequency and therefore the line will introduce frequency distortion.

Delay or Phase Distortion : For an applied voice-voltage wave the received waveform may not be identical with the input waveform at the sending end, since some frequency components will be delayed more than those of other frequencies. This phenomenon is known as delay or phase distortion.

The phase constant is

$$\beta = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

β is not a constant multiplied by ω and therefore the line will introduce delay distortion.

Frequency distortion is reduced in the transmission of high quality over wire lines by the use of equalizers at the line terminals.

Delay distortion is of relatively less importance to voice and music transmission. But it can be very serious for video transmission. This can be avoided by the use of co-axial cables.

1.7. THE DISTORTIONLESS LINE

If a line is to have neither frequency nor delay distortion, then attenuation factor α and the velocity of propagation v cannot be functions of frequency.

$$\text{If } v = \frac{\omega}{\beta}$$

β must be a direct function of frequency.

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

For β to be a direct function of frequency, the term

$(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2$ must be equal to $(RG + \omega^2 LC)^2$

$$R^2 G^2 + \omega^4 L^2 C^2 - 2\omega^2 LCRG + \omega^2 L^2 G + \omega^2 C^2 R^2 + 2\omega^2 LCRG$$

$$= R^2 G^2 + \omega^4 L^2 C^2 + 2\omega^2 LCRG$$

$$\omega^2 L^2 G^2 + \omega^2 C^2 R^2 = 2\omega^2 LCRG$$

$$\omega^2 L^2 G^2 + \omega^2 C^2 R^2 - 2\omega^2 LCRG = 0$$

$$(LG - CR)^2 = 0$$

$$LG = CR$$

$$\frac{R}{L} = \frac{G}{C}$$

This is the condition for distortionless line.

$$\begin{aligned}
 \text{Propagation constant } \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
 &= \sqrt{L \left(\frac{R}{L} + j\omega \right) C \left(\frac{G}{C} + j\omega \right)} \\
 &= \sqrt{LC} \sqrt{\left(\frac{R}{L} + j\omega \right) \left(\frac{G}{C} + j\omega \right)}
 \end{aligned}$$

$$\text{But } \frac{R}{L} = \frac{G}{C}$$

$$\gamma = \sqrt{LC} \left(\frac{R}{L} + j\omega \right)$$

$$\text{Then } \beta = \sqrt{\frac{\omega^2 LC - RG + RG + \omega^2 LC}{2}}$$

$$= \sqrt{\frac{2\omega^2 LC}{2}}$$

$$\beta = \omega \sqrt{LC}$$

Velocity of propagation is

$$v = \frac{\omega}{\beta}$$

$$v = \frac{1}{\sqrt{LC}}$$

This is the same velocity for all frequencies, thus eliminating delay distortion.

Attenuation factor

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

To make α is independent of frequency, the term $(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2$ is forced to be equal to $(RG + \omega^2 LC)^2$.

$$(LG - CR)^2 = 0$$

$$LG = CR$$

$$\boxed{\frac{L}{C} = \frac{R}{G}}$$

This will make α and the velocity independent of frequency simultaneously. To achieve this condition, it requires a very large value of L, since G is small.

The attenuation factor

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG + \omega^2 LC)^2}}{2}}$$

$$= \sqrt{\frac{RG - \omega^2 LC + RG + \omega^2 LC}{2}}$$

$$= \sqrt{\frac{2RG}{2}}$$

$$\alpha = \sqrt{RG}$$

It is independent of frequency, thus eliminating frequency distortion on the line.

The characteristic impedance Z_0 is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{L \left(\frac{R}{L} + j\omega \right)}{C \left(\frac{G}{C} + j\omega \right)}}$$

But $\frac{R}{L} = \frac{G}{C}$ for distortionless line.

$$\therefore Z_0 = \sqrt{\frac{L}{C}}$$

It is purely real and is independent of frequency.

1.8. TELEPHONE CABLE

In the telephone cable the wires are insulated with paper and twisted in pairs. This construction results in negligible values of inductance and conductance. Therefore $L\omega \ll R$ and $G \ll C\omega$.

$$Z = R + j\omega L \approx R$$

$$Y = G + j\omega C \approx j\omega C$$

Propagation constant

$$\gamma = \sqrt{ZY}$$

$$= \sqrt{j\omega RC}$$

$$= \sqrt{\frac{j2\omega RC}{2}}$$

$$\text{But } \gamma = \alpha + j\beta$$

$$\alpha + j\beta = (1 + j) \sqrt{\frac{\omega RC}{2}}$$

Equating real and imaginary parts

$$\alpha = \sqrt{\frac{\omega RC}{2}}$$

$$\beta = \sqrt{\frac{\omega RC}{2}}$$

$$\text{Velocity of propagation } v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega RC}{2}}} = \sqrt{\frac{2\omega}{RC}}$$

$$\text{The characteristic impedance } Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R}{j\omega C}} = \sqrt{\frac{R}{\omega C}} \angle -45^\circ$$

It is found that the propagation constant α and velocity of propagation v are functions of frequency. Thus, the higher frequencies are attenuated more and travel faster than the lower frequencies resulting in considerable frequency and delay distortion.

1.9. LOADING OF LINES

It is necessary to increase L/C ratio to achieve distortionless condition in a transmission line. This can be done by increasing the inductance of a transmission line. Increasing inductance by inserting inductances in series with line is termed as **loading** and such lines are called **loaded lines**. The lumped inductors, known as **loading coils** are placed at suitable intervals along the transmission line to increase the effective distributed inductance.

The effect of loading can be realised by comparing the unloading of a transmission line in the attenuation Vs frequency graph. Fig.1.5 shows that the loaded line offers a low attenuation when compared to the unloaded line only for limited range of frequencies.

The important aspect of loading coil design is that saturation and stray fields should be avoided. It should have a low resistance and should be in small size. In general toroidal cores are used for loading coils.

Types of Loading

The open wire lines have more inductance of their own and so have much less distortion than cable. Therefore, the loading practice is not applicable to open wires but it is restricted to cables only. There are three types of loading in practice. They are

- (a) Lumped loading
- (b) Continuous loading
- (c) Patch loading

(a) **Lumped loading**: The inductance of a transmission line can be increased by the introduction of loading coil at uniform intervals. This is called lumped loading. It acts as a low pass filter. So, it is applicable only for a limited range of frequency. The loading coils have an internal resistance R thus, increasing the total effective inductance increases R . Further hysteresis and eddy current losses which occur in the loading coils resulting in further apparent increase in R . Therefore, there is a practical limitation on the value of inductance that can be increased for the reduction of attenuation. Thus the loading coil should be carefully designed so that it will not introduce any distortion.

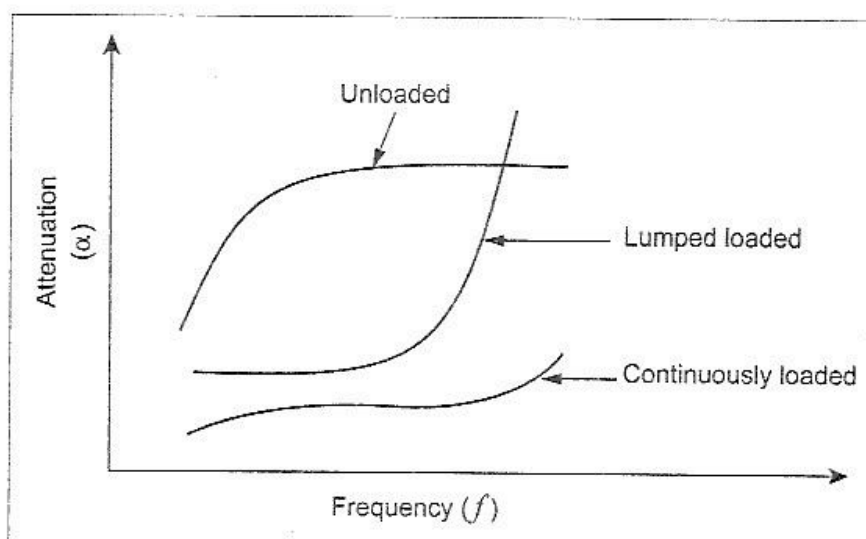


Fig. 1.5. Comparison of loaded and unloaded cable characteristics

(b) Continuous loading : A type of iron or some other magnetic material is wound on the transmission line (cable) to increase the permeability of the surrounding medium and thereby increase the inductance. It is a quite expensive method. Further eddy current and hysteresis losses in the magnetic material increases the primary constant R . Therefore, continuous loading is used only on ocean cables where lumped loading is difficult. The advantage of continuous loading over lumped loading is that attenuation factor α increases uniformly with increase in frequency.

(c) Patch loading : It employs sections of continuously loaded cable separated by sections of unloaded cable. The typical length for the section is normally a quarter kilometer. In this method the advantage of continuous loading is obtained and the cost is reduced considerably.

1.9.1. Inductance Loading of Telephone Cables

Distortionless line with distributed parameters is used to avoid the frequency and delay distortion experienced on telephone cables. It is necessary to increase the L/C to achieve distortionless condition $\frac{L}{C} = \frac{R}{G}$. Heaviside suggested that the inductance be increased and Pupin suggested that this increase in the inductance by lumped inductors spaced at intervals along the line. This use of inductance is called loading the line. The distributed loading is obtained by winding the cable with a high permeability steel tape such as permalloy in some submarine cables.

Consider an uniformly loaded cable with $G = 0$. Then,

$$Z = R + j\omega L$$

$$Y = j\omega C$$

$$[\because G = 0]$$

$$Z = \sqrt{R^2 + (L\omega)^2} \quad \tan^{-1} \left(\frac{L\omega}{R} \right)$$

$$= \sqrt{R^2 + (L\omega)^2} \left[\frac{\pi}{2} - \tan^{-1} \frac{R}{L\omega} \right]$$

Propagation constant $\gamma = \sqrt{ZY}$

$$\begin{aligned} &= \sqrt{\sqrt{R^2 + (L\omega)^2} \left[\frac{\pi}{2} - \tan^{-1} \frac{R}{L\omega} \right] \left(\omega C \left[\frac{\pi}{2} \right] \right)} \\ &= \sqrt{\omega C \sqrt{R^2 + (L\omega)^2} \left[\pi - \tan^{-1} \frac{R}{L\omega} \right]} \\ &= \sqrt{(\omega C)(L\omega)} \sqrt{1 + \frac{R^2}{(L\omega)^2}} \left[\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{L\omega} \right] \\ &= \omega \sqrt{LC} \sqrt{1 + \left(\frac{R}{L\omega} \right)^2} \left[\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{L\omega} \right] \end{aligned}$$

Since R is small with respect to $L\omega$, the term $\left(\frac{R}{L\omega} \right)$ is neglected.

$$\therefore \gamma = \omega \sqrt{LC} \left[\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{L\omega} \right]$$

$$\text{If } \theta = \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{L\omega}$$

$$\begin{aligned} \cos \theta &= \cos \left(\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{L\omega} \right) \\ &= \sin \left(\frac{1}{2} \tan^{-1} \frac{R}{L\omega} \right) \end{aligned}$$

For small angle,

$$\sin \theta \approx \tan \theta \approx \theta$$

so that

$$\cos \theta = \frac{R}{2L\omega}$$

Similarly,

$$\sin \theta = \sin \left(\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{L\omega} \right) = 1$$

Propagation constant $\gamma = \omega \sqrt{LC} (\cos \theta + j \sin \theta)$

$$= \omega \sqrt{LC} \left(\frac{R}{2L\omega} + j \right)$$

$$\gamma = \frac{R\sqrt{LC}}{2L} + j\omega \sqrt{LC}$$

$$= \frac{R}{2} \sqrt{\frac{C}{L}} + j\omega \sqrt{LC}$$

$$\therefore \text{Attenuation constant } \alpha = \frac{R}{2} \sqrt{\frac{C}{L}}$$

$$\text{Phase-shift } \beta = \omega \sqrt{LC}$$

$$\begin{aligned} \text{Velocity of propagation } v &= \frac{\omega}{\beta} \\ &= \frac{1}{\sqrt{LC}} \end{aligned}$$

It is noted that if $G = 0$ and $L\omega \gg R$, the attenuation and velocity are both independent of frequency and the loaded cable will be distortionless. Attenuation may be reduced by increasing L . Continuous (uniform) loading is expensive and achieves only a small increase in L per unit length. Lumped loading is preferred for cables.

Campbell's Equation

An analysis for the performance of a line loaded at uniform intervals can be obtained by considering a symmetrical section of line from the centre of one loading coil to the centre of the next coil. The section of line may be replaced with an equivalent T section having symmetrical series arms as shown in Fig.1.6. The series arm of T section including loading coil is given by

$$\frac{Z_1'}{2} = \frac{Z_c}{2} + \frac{Z_1}{2} \quad [\text{From the fig.}]$$

where $\frac{Z_1}{2}$ is the series arm of T section.

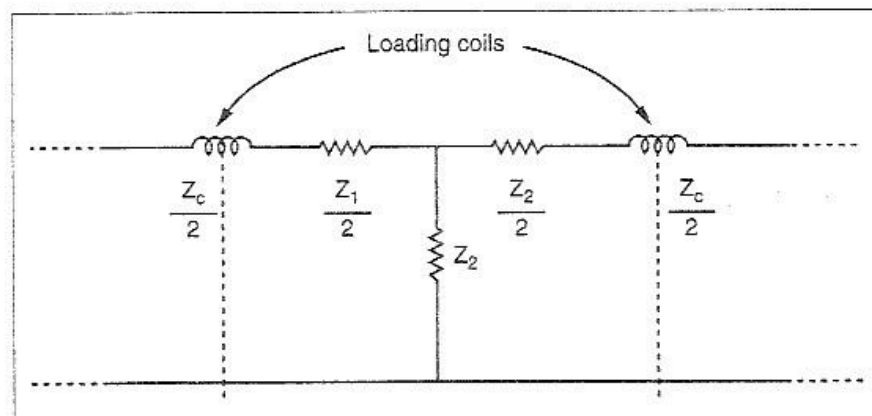


Fig. 1.6. Equivalent T section for part of a line between two lumped loading coils

$$\frac{Z_1}{2} = Z_o \tanh \frac{\gamma l}{2}$$

$$\therefore \frac{Z_1'}{2} = \frac{Z_c}{2} + Z_o \tanh \frac{\gamma l}{2}$$

where l is the distance between two loading coils.

The shunt Z_2 arm of the equivalent T section is

$$Z_2 = \frac{Z_0}{\sinh \gamma l}$$

For loaded T section

$$\begin{aligned} \cosh \gamma' l &= 1 + \frac{Z_1'}{2 Z_2} \\ &= 1 + \frac{\frac{Z_c}{2} + Z_0 \tanh \frac{\gamma l}{2}}{\frac{Z_0}{\sinh \gamma l}} \end{aligned}$$

$$\text{But } \tanh \frac{\gamma l}{2} = \frac{\cosh \gamma l - 1}{\sinh \gamma l}$$

Substituting this value in above equation

$$\begin{aligned} \therefore \cosh \gamma' l &= 1 + \frac{\frac{Z_c}{2} + Z_0 \frac{\cosh \gamma l - 1}{\sinh \gamma l}}{\frac{Z_0}{\sinh \gamma l}} \\ &= 1 + \frac{\frac{Z_c}{2} \sinh \gamma l + Z_0 (\cosh \gamma l - 1)}{Z_0} \\ &= 1 + \frac{Z_c}{2 Z_0} \sinh \gamma l + \cosh \gamma l - 1 \\ \cosh \gamma' l &= \frac{Z_c}{2 Z_0} \sinh \gamma l + \cosh \gamma l \end{aligned}$$

This equation is called as Campbell's equation and it is used to determine the value of γ' of a line section consisting of partially lumped and partially distributed elements. For a cable Z_2 is capacitance and the cable capacitance and lumped inductances appear similar to the circuit of the low pass filter. It is found that for frequencies below cutoff, the attenuation is reduced, but the cut-off attenuation is increased (as a result of filter action). In practice, pure distortionless line is not obtained by loading, because R and L are to some extent functions of frequency. Eddy current losses are more in these coils. However, there is a major improvement in the loaded cable over the unloaded cable for a reasonable frequency range.

1.10. OPEN CIRCUITED AND SHORT CIRCUITED LINES

The expressions for voltage and current at the sending end of a transmission line of length ' l ' are given by

$$V_S = V_R \left[\cosh \sqrt{ZY} l + \frac{Z_0}{Z_R} \sinh \sqrt{ZY} l \right]$$

$$I_S = I_R \left[\cosh \sqrt{ZY} l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY} l \right]$$

The input impedance of a transmission line is given by

$$\begin{aligned} Z_S &= \frac{V_S}{I_S} \\ &= \frac{V_R \left[\cosh \sqrt{ZY} l + \frac{Z_0}{Z_R} \sinh \sqrt{ZY} l \right]}{I_R \left[\cosh \sqrt{ZY} l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY} l \right]} \\ &= \frac{V_R}{I_R} \frac{Z_0}{Z_R} \frac{(Z_R \cosh \gamma l + Z_0 \sinh \gamma l)}{(Z_0 \cosh \gamma l + Z_R \sinh \gamma l)} \\ &= Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right) \quad \left[\because Z_R = \frac{V_R}{I_R} \right] \\ Z_S &= Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right) \end{aligned}$$

If short circuited, the receiving end impedance is zero.

$$\text{i.e., } Z_R = 0$$

$$\therefore Z_{sc} = Z_0 \left(\frac{Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l} \right)$$

Short circuited impedance

$$Z_{sc} = Z_0 \tanh \gamma l$$

If open circuited, the receiving end impedance is infinite.

$$\text{i.e., } Z_R = \infty$$

Input impedance of transmission line can be written as

$$Z_S = Z_0 \left[\frac{\cosh \gamma l + \frac{Z_0}{Z_R} \sinh \gamma l}{\frac{Z_0}{Z_R} \cosh \gamma l + \sinh \gamma l} \right]$$

Applying $Z_R = \infty$

$$\text{Then } Z_{oc} = Z_0 \left[\frac{\cosh \gamma l}{\sinh \gamma l} \right]$$

The open circuited impedance

$$Z_{oc} = Z_0 \coth \gamma l$$

By multiplying open circuited impedance and short circuited impedances

$$\begin{aligned} Z_{oc} Z_{sc} &= Z_0^2 \tanh \gamma l \coth \gamma l \\ &= Z_0^2 \end{aligned}$$

The characteristic impedance is given by

$$Z_0 = \sqrt{Z_{oc} Z_{sc}}$$

By dividing short circuited impedance by open circuited impedance.

$$\begin{aligned} \frac{Z_{sc}}{Z_{oc}} &= \frac{Z_0 \tanh \gamma l}{Z_0 \coth \gamma l} \\ &= \tanh^2 \gamma l \\ \tanh \gamma l &= \sqrt{\frac{Z_{sc}}{Z_{oc}}} \\ \gamma l &= \tanh^{-1} \sqrt{\frac{Z_{sc}}{Z_{oc}}} \end{aligned}$$

1.11. REFLECTION

When the load impedance is not equal to the characteristic impedance of transmission line, reflection takes place.

The expressions for voltage and current on the transmission line are

$$V = \frac{V_R}{2} \left[\left(1 + \frac{Z_0}{Z_R} \right) e^{\sqrt{ZY}x} + \left(1 - \frac{Z_0}{Z_R} \right) e^{-\sqrt{ZY}x} \right]$$

$$I = \frac{I_R}{2} \left[\left(1 + \frac{Z_R}{Z_0} \right) e^{\sqrt{ZY}x} + \left(1 - \frac{Z_R}{Z_0} \right) e^{-\sqrt{ZY}x} \right]$$

or

$$V = \frac{V_R}{2} \left[\frac{Z_R + Z_0}{Z_R} e^{\sqrt{ZY}x} + \frac{Z_R - Z_0}{Z_R} e^{-\sqrt{ZY}x} \right]$$

$$I = \frac{I_R}{2} \left[\frac{Z_R + Z_0}{Z_0} e^{\sqrt{ZY}x} - \frac{Z_R - Z_0}{Z_0} e^{-\sqrt{ZY}x} \right]^{133}$$

or

$$V = \frac{V_R (Z_R + Z_0)}{2 Z_R} \left[e^{\gamma x} + \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\gamma x} \right]$$

$$I = \frac{I_R (Z_R + Z_0)}{2 Z_R} \left[e^{\gamma x} - \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\gamma x} \right]$$

$$[\because \gamma = \sqrt{ZY}]$$

If the transmission line is not terminated with the characteristic impedance *i.e.*, $Z_R \neq Z_0$ (mismatch) the above expressions for voltage and current exist. It consists of two waves, one is moving in the forward (positive x) direction which is called incident wave and the other is moving in the opposite (negative x) direction which is called reflected ray. The term varying with $e^{\gamma x}$ represents a wave progressing from the sending end towards the receiving end and the amplitude decreasing with increased distance. The term varying with $e^{-\gamma x}$ represents a wave progressing from the receiving end towards the sending end, decreasing in amplitude with increased distance.

If the transmission line is terminated with characteristic impedance *i.e.*, $Z_R = Z_0$ (properly matched) then the voltage and current expressions are

$$V = V_R e^{\gamma x}$$

$$I = I_R e^{\gamma x}$$

The incident wave moves only in forward (positive x) direction. There is no reflected wave in the opposite direction.

1.11.1. Reflection Coefficient

Reflection coefficient is defined as the ratio of the reflected voltage to the incident voltage at the receiving end of the line.

$$K = \frac{\text{Reflected voltage at load}}{\text{Incident voltage at load}} = \frac{V_R}{V_S}$$

The equation for the voltage of a transmission line is

$$V = \frac{V_R (Z_R + Z_0)}{2 Z_R} \left[e^{\gamma x} + \left(\frac{Z_R - Z_0}{Z_R + Z_0} \right) e^{-\gamma x} \right]$$

$$V = \frac{V_R (Z_R + Z_0)}{2 Z_R} e^{\gamma x} + \frac{V_R (Z_R - Z_0)}{2 Z_R} e^{-\gamma x}$$

The first term ($e^{\gamma x}$) represents incident wave, whereas the second term ($e^{-\gamma x}$) represents the reflected wave. The ratio of amplitude of the reflected wave voltage to the amplitude of the incident wave voltage is nothing but reflection coefficient.

$$K = \frac{\frac{V_R (Z_R - Z_0)}{2 Z_R}}{\frac{V_R (Z_R + Z_0)}{2 Z_R}} = \frac{Z_R - Z_0}{Z_R + Z_0}$$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0}$$

It is also defined as in terms of the ratio of the reflected current to the incident current. But it is negative.

$$-K = \frac{\text{Reflected current at load}}{\text{Incident current at load}} = \frac{I_R}{I_S}$$

If the transmission line is terminated by its characteristic impedance ($Z_R = Z_0$), the reflection coefficient becomes zero.

1.11.2. Reflection Factor and Reflection Loss

Consider a transmission line with a voltage source V_S and its impedance Z_1 and load impedance Z_2 as shown in Fig.1.7. If Z_2 is not equal to Z_1 , reflection takes place. The power delivered to the load is less than that with impedance matching. Reflection results in power loss. This loss is known as reflection loss.

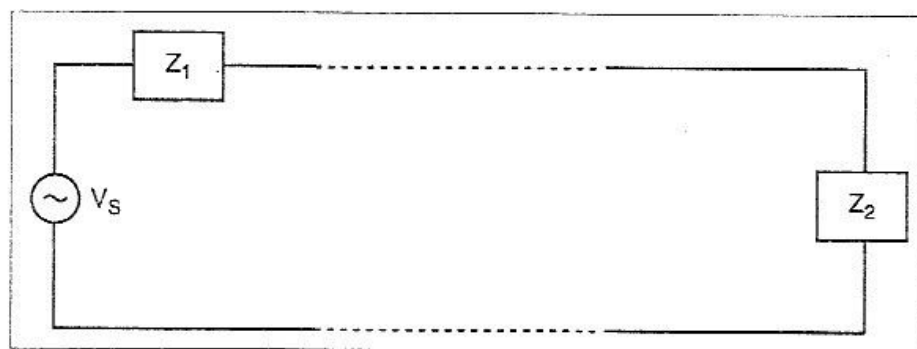


Fig. 1.7. Transmission line with voltage source V_S and impedance Z_1

Image matching between the impedances Z_1 and Z_2 can be obtained by inserting an ideal transformer and a phase shifting network between Z_1 and Z_2 . If I_1 and I_2 be the currents in the primary and secondary of the transformer respectively, the current ratio of the transformer is given by

$$\frac{I_2}{I_1} = \sqrt{\frac{Z_1}{Z_2}}$$

Z_2 may be adjusted to that of Z_1 by choosing the proper transformation ratio and phase angle. Z_2 is the image impedance of Z_1 . The current through the source is

$$I_1 = \frac{V_S}{2 Z_1}$$

The current flow in the secondary of the transformer under image impedance matching is

$$I_2' = I_1 \sqrt{\frac{Z_1}{Z_2}} = \frac{V}{2 Z_1} \sqrt{\frac{Z_1}{Z_2}} = \frac{V_S}{2 \sqrt{Z_1 Z_2}}$$

The current in the load impedance Z_2 without image matching.

$$|I_2| = \frac{|V_S|}{|Z_1 + Z_2|}$$

The ratio of the current actually flowing in the load to that which might flow under matched condition is known as **reflection factor**.

$$\left| \frac{I_2}{I_2'} \right| = \frac{\frac{|V_S|}{|Z_1 + Z_2|}}{\frac{|V_S|}{|2 \sqrt{Z_1 Z_2}|}}$$

$$k = \left| \frac{2 \sqrt{Z_1 Z_2}}{Z_1 + Z_2} \right|$$

The reflection factor indicates the change in current in the load due to reflection at the mismatched junction.

The **reflection loss** is the reciprocal of the reflection factor in nepers or dB.

$$\begin{aligned} \text{Reflection loss} &= \ln \frac{1}{k} \\ &= \ln \left| \frac{Z_1 + Z_2}{2 \sqrt{Z_1 Z_2}} \right| \text{ nepers} \\ &= 20 \log \left| \frac{Z_1 + Z_2}{2 \sqrt{Z_1 Z_2}} \right| \text{ dB} \end{aligned}$$

1.12. T AND π SECTIONS EQUIVALENT TO LINES

A T section is shown in Fig.1.8 with two ports 1, 1 and 2, 2.

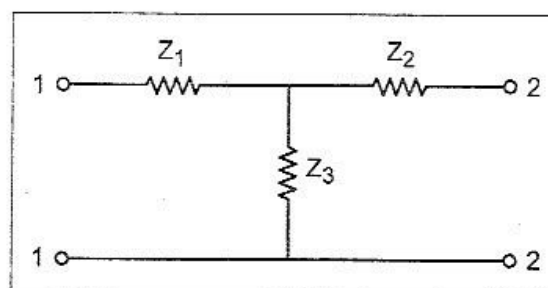


Fig. 1.8. T section network

Impedance measurements may be made at any port with the other port opened or shorted.

Let Z_{1OC} be the impedance at port 1 when port 2 is open circuited.

Z_{1SC} be the impedance at port 1 when port 2 is short circuited.

Z_{2OC} be the impedance at port 2 when port 1 is open circuited.

Z_{2SC} be the impedance at port 2 when port 1 is short circuited.

$$Z_{1OC} = Z_1 + Z_3$$

$$Z_{1SC} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$Z_{2OC} = Z_2 + Z_3$$

$$Z_{2SC} = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3}$$

By solving these equations, the values of Z_1 , Z_2 and Z_3 are determined.

$$\begin{aligned} Z_{1OC} - Z_{1SC} &= Z_3 - \frac{Z_2 Z_3}{Z_2 + Z_3} \\ &= \frac{Z_3 Z_2 + Z_3^2 - Z_2 Z_3}{Z_2 + Z_3} \end{aligned}$$

$$= \frac{Z_3^2}{Z_2 + Z_3}$$

$$= \frac{Z_3^2}{Z_{2OC}}$$

$$[\because Z_2 + Z_3 = Z_{2OC}]$$

$$Z_3^2 = Z_{2OC} (Z_{1OC} - Z_{1SC})$$

$$Z_3 = \pm \sqrt{Z_{2OC} (Z_{1OC} - Z_{1SC})}$$

Taking the positive value,

$$Z_3 = \sqrt{Z_{2OC} (Z_{1OC} - Z_{1SC})}$$

$$Z_1 = Z_{1OC} - Z_3$$

$$[\because Z_{1OC} = Z_1 + Z_3]$$

$$= Z_{1OC} - \sqrt{Z_{2OC} (Z_{1OC} - Z_{1SC})}$$

$$Z_2 = Z_{2OC} - Z_3$$

$$[\because Z_{2OC} = Z_2 + Z_3]$$

$$= Z_{2OC} - \sqrt{Z_{2OC} (Z_{1OC} - Z_{1SC})}$$

$$Z_1 = Z_{1OC} - \sqrt{Z_{2OC} (Z_{1OC} - Z_{1SC})}$$

$$Z_2 = Z_{2OC} - \sqrt{Z_{2OC} (Z_{1OC} - Z_{1SC})}$$

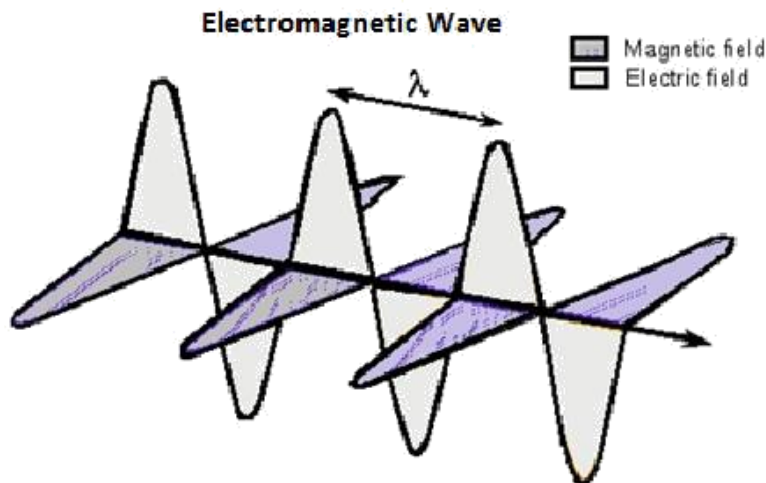
UNIT - V**Transmission Lines – II**

- SC and OC Lines
- Input Impedance Relations
- Reflection Coefficient
- VSWR
- $\lambda/4$, $\lambda/2$, $\lambda/8$ Lines - Impedance Transformations
- Smith Chart - Configuration and Applications,
- Single Stub Matching
- Illustrative Problems.

This means, more the current flows towards the surface of the conductor, it flows less towards the center, which is known as the **Skin Effect**.

Inductance

In an AC transmission line, the current flows sinusoidally. This current induces a magnetic field perpendicular to the electric field, which also varies sinusoidally. This is well known as Faraday's law. The fields are depicted in the following figure.



This varying magnetic field induces some EMF into the conductor. Now this induced voltage or EMF flows in the opposite direction to the current flowing initially. This EMF flowing in the opposite direction is equivalently shown by a parameter known as **Inductance**, which is the property to oppose the shift in the current.

It is denoted by "**L**". The unit of measurement is "**Henry H**".

Conductance

There will be a leakage current between the transmission line and the ground, and also between the phase conductors. This small amount of leakage current generally flows through the surface of the insulator. Inverse of this leakage current is termed as **Conductance**. It is denoted by "**G**".

The flow of line current is associated with inductance and the voltage difference between the two points is associated with capacitance. Inductance is associated with the magnetic field, while capacitance is associated with the electric field.

Capacitance

The voltage difference between the **Phase conductors** gives rise to an electric field between the conductors. The two conductors are just like parallel plates and the air in between them becomes dielectric. This pattern gives rise to the capacitance effect between the conductors.

Characteristic Impedance

If a uniform lossless transmission line is considered, for a wave travelling in one direction, the ratio of the amplitudes of voltage and current along that line, which has no reflections, is called as **Characteristic impedance**.

It is denoted by Z_0

$$Z_0 = \sqrt{\frac{\text{voltage wave value}}{\text{current wave value}}}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

For a lossless line, $R_0 = \sqrt{\frac{L}{C}}$

Where L & C are the inductance and capacitance per unit lengths.

Impedance Matching

To achieve maximum power transfer to the load, impedance matching has to be done. To achieve this impedance matching, the following conditions are to be met.

The resistance of the load should be equal to that of the source.

$$R_L = R_S$$

The reactance of the load should be equal to that of the source but opposite in sign.

$$X_L = -X_S$$

Which means, if the source is inductive, the load should be capacitive and vice versa.

Reflection Co-efficient

The parameter that expresses the amount of reflected energy due to impedance mismatch in a transmission line is called as **Reflection coefficient**. It is indicated by ρ rho.

It can be defined as "the ratio of reflected voltage to the incident voltage at the load terminals".

$$\rho = \frac{\text{reflected voltage}}{\text{incident voltage}} = \frac{V_r}{V_i} \text{ at load terminals}$$

If the impedance between the device and the transmission line don't match with each other, then the energy gets reflected. The higher the energy gets reflected, the greater will be the value of ρ reflection coefficient.

Voltage Standing Wave Ratio $VSWR$

The standing wave is formed when the incident wave gets reflected. The standing wave which is formed, contains some voltage. The magnitude of standing waves can be measured in terms of standing wave ratios.

The ratio of maximum voltage to the minimum voltage in a standing wave can be defined as Voltage Standing Wave Ratio $VSWR$. It is denoted by " S ".

$$S = \frac{|V_{max}|}{|V_{min}|} \quad 1 \leq S \leq \infty$$

VSWR describes the voltage standing wave pattern that is present in the transmission line due to phase addition and subtraction of the incident and reflected waves.

Hence, it can also be written as

$$S = \frac{1 + \rho}{1 - \rho}$$

The larger the impedance mismatch, the higher will be the amplitude of the standing wave. Therefore, if the impedance is matched perfectly,

$$V_{max} : V_{min} = 1 : 1$$

Hence, the value for VSWR is unity, which means the transmission is perfect.

Efficiency of Transmission Lines

The efficiency of transmission lines is defined as the ratio of the output power to the input power.

$$\% \text{ efficiency of transmission line } \eta = \frac{\text{Power delivered at reception}}{\text{Power sent from the transmission end}} \times 100$$

Voltage Regulation

Voltage regulation is defined as the change in the magnitude of the voltage between the sending and receiving ends of the transmission line.

$$\% \text{ voltage regulation} = \frac{\text{sending end voltage} - \text{receiving end voltage}}{\text{sending end voltage}} \times 100$$

Losses due to Impedance Mismatch

The transmission line, if not terminated with a matched load, occurs in losses. These losses are many types such as attenuation loss, reflection loss, transmission loss, return loss, insertion loss, etc.

Attenuation Loss

The loss that occurs due to the absorption of the signal in the transmission line is termed as Attenuation loss, which is represented as

$$\text{Attenuation loss(dB)} = 10 \log_{10} \left[\frac{E_i - E_r}{E_t} \right]$$

Where

- E_i = the input energy
- E_r = the reflected energy from the load to the input
- E_t = the transmitted energy to the load

Reflection Loss

The loss that occurs due to the reflection of the signal due to impedance mismatch of the transmission line is termed as Reflection loss, which is represented as

$$\text{Reflection loss(dB)} = 10 \log_{10} \left[\frac{E_i}{E_i - E_r} \right]$$

Where

- E_i = the input energy
- E_r = the reflected energy from the load

Transmission Loss

The loss that occurs while transmission through the transmission line is termed as Transmission loss, which is represented as

$$\text{Transmission loss(dB)} = 10 \log_{10} \frac{E_i}{E_t}$$

Where

- E_i = the input energy
- E_t = the transmitted energy

Return Loss

The measure of the power reflected by the transmission line is termed as Return loss, which is represented as

$$\text{Return loss(dB)} = 10 \log_{10} \frac{E_i}{E_r}$$

Where

- E_i = the input energy
- E_r = the reflected energy

Insertion Loss

The loss that occurs due to the energy transfer using a transmission line compared to energy transfer without a transmission line is termed as Insertion loss, which is represented as

$$\text{Insertion loss(dB)} = 10 \log_{10} \frac{E_1}{E_2}$$

Where

- E_1 = the energy received by the load when directly connected to the source, without a transmission line.
- E_2 = the energy received by the load when the transmission line is connected between the load and the source.

Stub Matching

If the load impedance mismatches the source impedance, a method called "Stub Matching" is sometimes used to achieve matching.

The process of connecting the sections of open or short circuit lines called **stubs** in the shunt with the main line at some point or points, can be termed as **Stub Matching**.

At higher microwave frequencies, basically two stub matching techniques are employed.

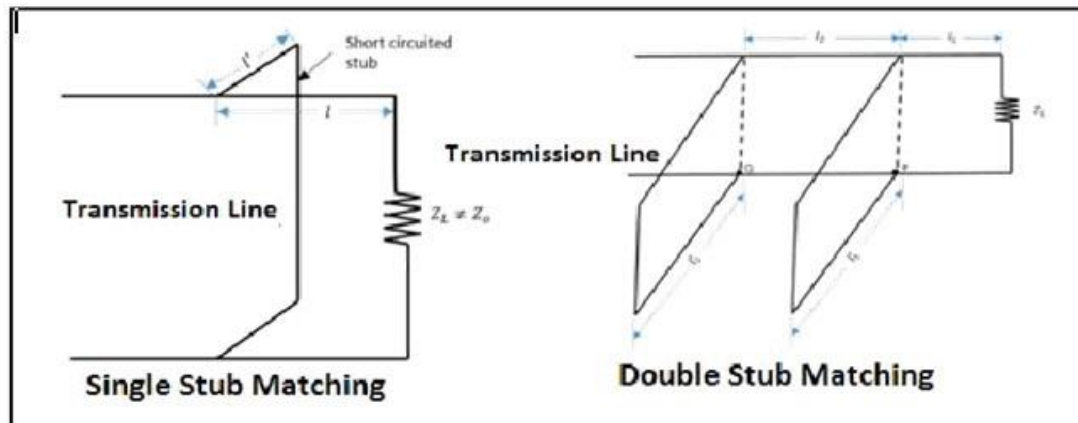
Single Stub Matching

In Single stub matching, a stub of certain fixed length is placed at some distance from the load. It is used only for a fixed frequency, because for any change in frequency, the location of the stub has to be changed, which is not done. This method is not suitable for coaxial lines.

Double Stub Matching

In double stub matching, two stubs of variable length are fixed at certain positions. As the load changes, only the lengths of the stubs are adjusted to achieve matching. This is widely used in laboratory practice as a single frequency matching device.

The following figures show how the stub matchings look.



Transmission Lines – Smith Chart & Impedance Matching (Intensive Reading)

1 The Smith Chart

Transmission line calculations – such as the determination of input impedance using equation (4.30) and the reflection coefficient or load impedance from equation (4.32) – often involves tedious manipulation of complex numbers. This tedium can be alleviated using a graphical method of solution. The best known and most widely used graphical chart is the Smith chart. The Smith chart is a circular plot with a lot of interlaced circles on it. When used correctly, impedance matching can be performed without any computation. The only effort required is the reading and following of values along the circles.

The Smith chart is a polar plot of the complex reflection coefficient, or equivalently, a graphical plot of normalized resistance and reactance functions in the reflection-coefficient plane. To understand how the Smith chart for a lossless transmission line is constructed, examine the voltage reflection coefficient of the load impedance defined by

$$\Gamma_L = \frac{V_{\text{refl}}}{V_{\text{inc}}} = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{\text{re}} + j\Gamma_{\text{im}} \quad (1)$$

where Γ_{re} and Γ_{im} are the real and imaginary parts of the complex reflection coefficient Γ_L . The characteristic impedance Z_0 is often a constant and a real industry normalized value, such as 50 Ω , 75 Ω , 100 Ω , and 600 Ω . We can then define the normalised load impedance by

$$z_L = Z_L / Z_0 = (R + jX) / Z_0 = r + jx. \quad (2)$$

With this simplification, we can rewrite the reflection coefficient formula in (1) as

$$\Gamma_L = \Gamma_{\text{re}} + j\Gamma_{\text{im}} = \frac{(Z_L - Z_0) / Z_0}{(Z_L + Z_0) / Z_0} = \frac{z_L - 1}{z_L + 1}. \quad (3)$$

The inverse relation of (3) is

$$z_L = \frac{1 + \Gamma_L}{1 - \Gamma_L} = \frac{1 + |\Gamma_L| e^{j\theta}}{1 - |\Gamma_L| e^{j\theta}} \quad (4)$$

or

$$r + jx = \frac{(1 + \Gamma_{\text{re}}) + j\Gamma_{\text{im}}}{(1 - \Gamma_{\text{re}}) - j\Gamma_{\text{im}}} \quad (5)$$

Multiplying both the numerator and the denominator of (5) by the complex conjugate of the denominator and separating the real and imaginary parts, we obtain

$$r = \frac{1 - \Gamma_{re}^2 - \Gamma_{im}^2}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2} \quad (6)$$

and

$$x = \frac{2\Gamma_{im}^2}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2} \quad (7)$$

Equation (6) can be rearranged
as

$$\Gamma_{re}^2 - \frac{r}{1+r} + \Gamma_{im}^2 = \frac{1-r}{1+r} \quad (8)$$

This equation is a relationship in the form of a parametric equation $(x - a)^2 + (y - b)^2 = R^2$ in the complex plane $(\Gamma_{re}, \Gamma_{im})$ of a circle centred at the coordinates $(r + 1, 0)$ and having a

$$\text{radius of } 1$$

Different values of r yield circles of different radii with centres at different

$$r + 1$$

positions on the Γ_{re} -axis. The following properties of the r -circles are noted:

- The centres of all r -circles lie on the Γ_{re} -axis.
- The circle where there is no resistance ($r = 0$) is the largest. It is centred at the origin and has a radius of 1.
- The r -circles become progressively smaller as r increases from 0 to ∞ , ending at the $(\Gamma_{re} = 1, \Gamma_{im} = 0)$ point for an open circuit.
- All the r -circles pass through the point $(\Gamma_{re} = 1, \Gamma_{im} = 0)$.

See Figure 1 for further details.

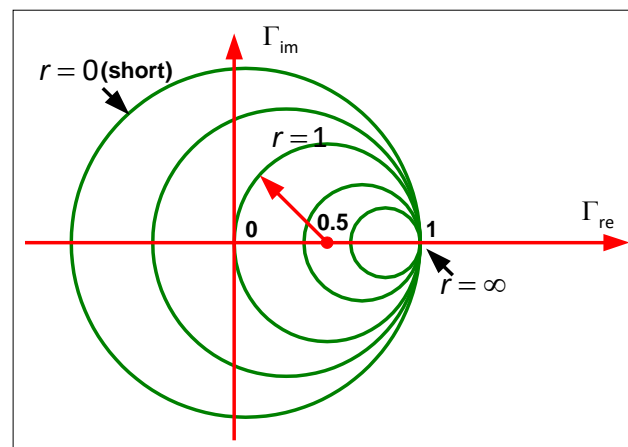


Figure 1: The r -circles in the complex plane $(\Gamma_{re}, \Gamma_{im})$.

Similarly, (7) can be rearranged as

$$(\Gamma_{re} - 1)^2 + \Gamma_{im}^2 = 1 \quad (9)$$

Again, (9) is a parametric equation of the type $(x - a)^2 + (y - b)^2 = R^2$ in the complex plane

(Γ_r, Γ_i) of a circle centred at the coordinates $(1, 1/x)$ and having a radius of $1/x$. Different values of x yield circles of different radii with centres at different positions on the $\Gamma_{re} = 1$ line.

The following properties of the x -circles are noted:

- The centres of all x -circles lie on the $\Gamma_{re} = 1$ line; those for $x > 0$ (inductive reactance) lie above the Γ_{re} -axis, and those for $x < 0$ lie below the Γ_{re} -axis.
- The $x = 0$ circle becomes the Γ_{re} -axis.
- The x -circles become progressively smaller as $|x|$ increases from 0 to ∞ , ending at the ($\Gamma_{re} = 1, \Gamma_{im} = 0$) point for an open circuit.
- All the x -circles pass through the point ($\Gamma_{re} = 1, \Gamma_{im} = 0$).

See Figure 2 for further details.

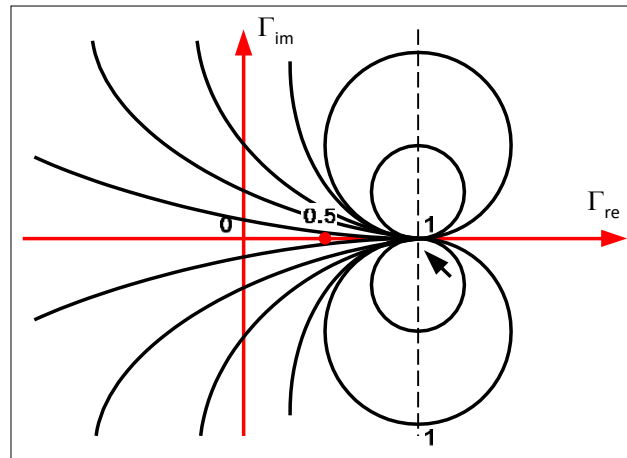


Figure 2: The x-circles in the complex plane (Γ_{re} , Γ_{im}).

To complete the Smith chart, the two circles' families are superimposed. The Smith chart therefore becomes a chart of r - and x -circles in the $(\Gamma_{re}, \Gamma_{im})$ -plane for $|\Gamma| \leq 1$. The

intersection of an r -circle and an x -circle defines a point which represents a normalized load

impedance $z_L = r + jx$. The actual load impedance is $Z_L = Z_0 z_L = Z_0 (r + jx)$. As an

illustration, the impedance $Z_L = 85 + j30$ in a $Z_0 = 50 \Omega$ -system is represented by the point P in Figure 3. Here $z_L = 1.7 + j0.6$ at the intersection of the $r = 1.7$ and the $x = 0.6$ circles. Values for Γ_{re} and Γ_{im} may then be obtained from the projections onto the horizontal and

vertical axes (see Figure 4). These are approximately given by $\Gamma_{re} \approx 0.3$ and $\Gamma_{im} \approx 0.16$.

Point P_{sc} at $(\Gamma_{re} = -1, \Gamma_{im} = 0)$ corresponds to $r = 0$ and $x = 0$ and therefore represents a

short-circuit. Point P_{oc} at $(\Gamma_{re} = 1, \Gamma_{im} = 0)$ corresponds to an infinite impedance therefore

represents an open circuit.

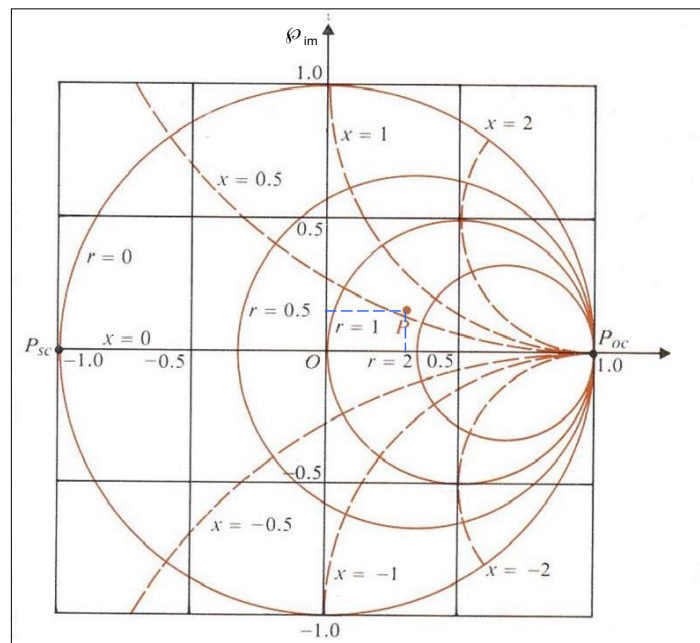


Figure 3: Smith chart with rectangular coordinates.

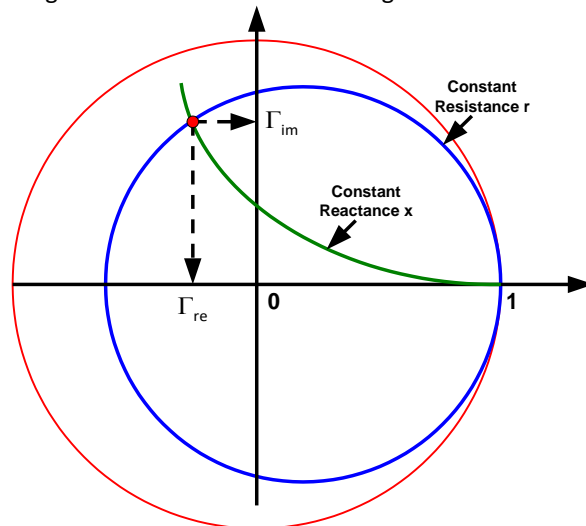


Figure 4: Direct extraction of the reflection coefficient $\Gamma = \Gamma_{re} + j\Gamma_{im}$ along the horizontal and vertical axes.

Instead of having a Smith chart marked with Γ_{re} and Γ_{im} marked in rectangular coordinates, the same chart can be marked in polar coordinates, so that every point in the Γ -plane is specified by a magnitude $|\Gamma|$ and a phase angle θ . This is illustrated in Figure 5, where several $|\Gamma|$ -circles are shown in dashed lines and some θ -angles are marked around the $|\Gamma| = 1$ circle. The $|\Gamma|$ -circles are normally not shown on commercially available Smith charts, but once the point representing a certain $z_L = r + jx$ is located, it is simply a matter of

drawing a circle centred at the origin through the point. The ratio of the distance to the point and the radius to the edge of the chart is equal to the magnitude of Γ of the load reflection coefficient, and the angle that a line to that point makes with the real axis represents θ . If, for

example the point

$$\begin{array}{lll} Z_2 = 75 - j100 \wedge & Z_3 = j200 \wedge & Z_4 = 150 \wedge \\ Z_6 = 0 \text{ (a short circuit)} & Z_7 = 50 \wedge & Z_8 = 184 - j900 \\ & & \wedge \end{array}$$

$$\begin{array}{l} Z_1 = 100 + j50 \wedge \\ Z_5 = \infty \text{ (an open} \\ \text{circuit)} \end{array}$$

The normalized impedances shown below are plotted in Figure 6.

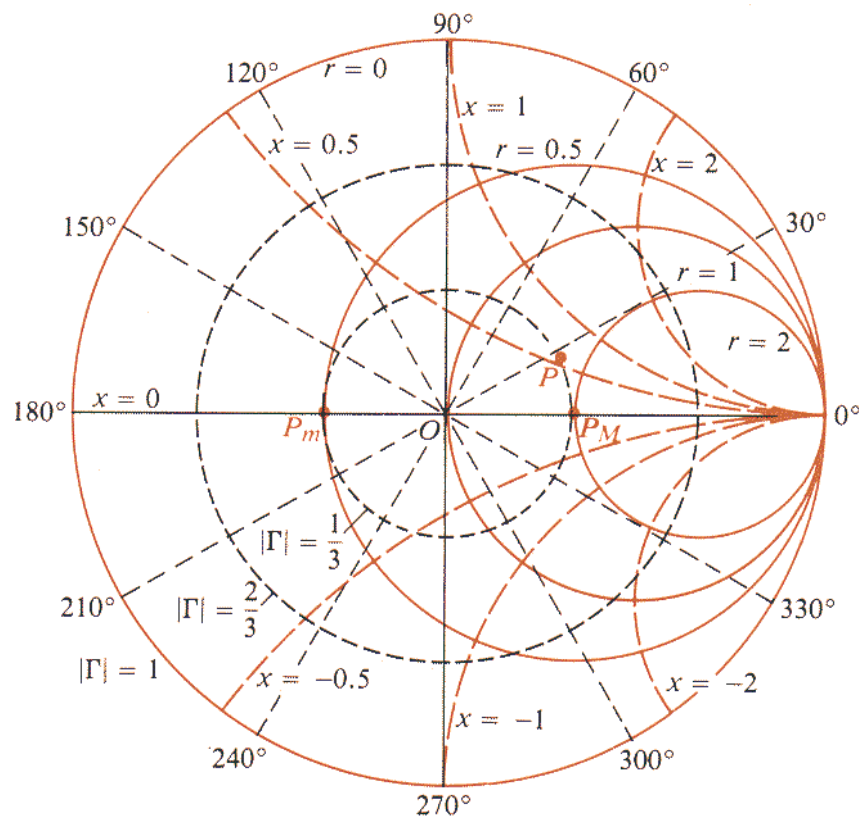
$$\begin{array}{llll} z_1 = 2 + j & z_2 = 1.5 - j2 & z_3 = j4 & z_4 = 3 \\ z_5 = \infty & z_6 = 0 & z_7 = 1 & z_8 = 3.68 - j18 \end{array}$$

It is also possible to directly extract the reflection coefficient Γ on the Smith chart of Figure 6. Once the impedance point is plotted (the intersection point of a constant resistance circle and

$z_L = 1.7 + j0.6$ is marked on the Smith chart at point P , we find that

$$|\Gamma_L| = 1/3 \text{ and } \theta = 28^\circ.$$

Each r -circle intersects the real axis at two points. In Figure 5 we designate the point on the positive real axis as P_M and on the negative real axis as P_m . Since $x = 0$ along the real axis, both these points represent situations of a purely resistive load, $Z_L = R_L$. Obviously, $R_L > Z_0$ at P_M where $r > 1$, and $R_L < Z_0$ at P_m where $r < 1$. Since $S = R_L / Z_0$ for $R_L > Z_0$, the value of the r -circle passing through the point P_M is numerically equal to the standing wave ratio. For the



example where $z_L = 1.7 + j0.6$, we find that $r = 2$ at P_M , so that $S = r = 2$.

Figure 5: Smith chart in polar coordinates.

Example 1:

Consider a characteristic impedance of 50Ω with the following impedances:

of a constant reactance circle), simply read the rectangular coordinates projection on the horizontal and vertical axis. This will give Γ_{re} , the real part of the reflection coefficient, and

Γ_{im} , the imaginary part of the reflection coefficient. Alternatively, the reflection coefficient may be obtained in polar form by using the scales provided on the commercial Smith chart.

$$\begin{array}{llll}
 \Gamma_1 = 0.4 + 0.2j & \Gamma_2 = 0.51 - 0.4j & \Gamma_3 = 0.875 + 0.48j & \Gamma_4 = 0.5 \\
 = 0.45 \angle 27^\circ & = 0.65 \angle -38^\circ & = 0.998 \angle 29^\circ & = 0.5 \angle 0^\circ \\
 \Gamma_5 = 1 & \Gamma_6 = -1 & \Gamma_7 = 0 & \Gamma_8 = 0.96 - 0.1j \\
 = 1 \angle 0^\circ & = 1 \angle 180^\circ & = 0 & = 0.97 \angle -6^\circ
 \end{array}$$

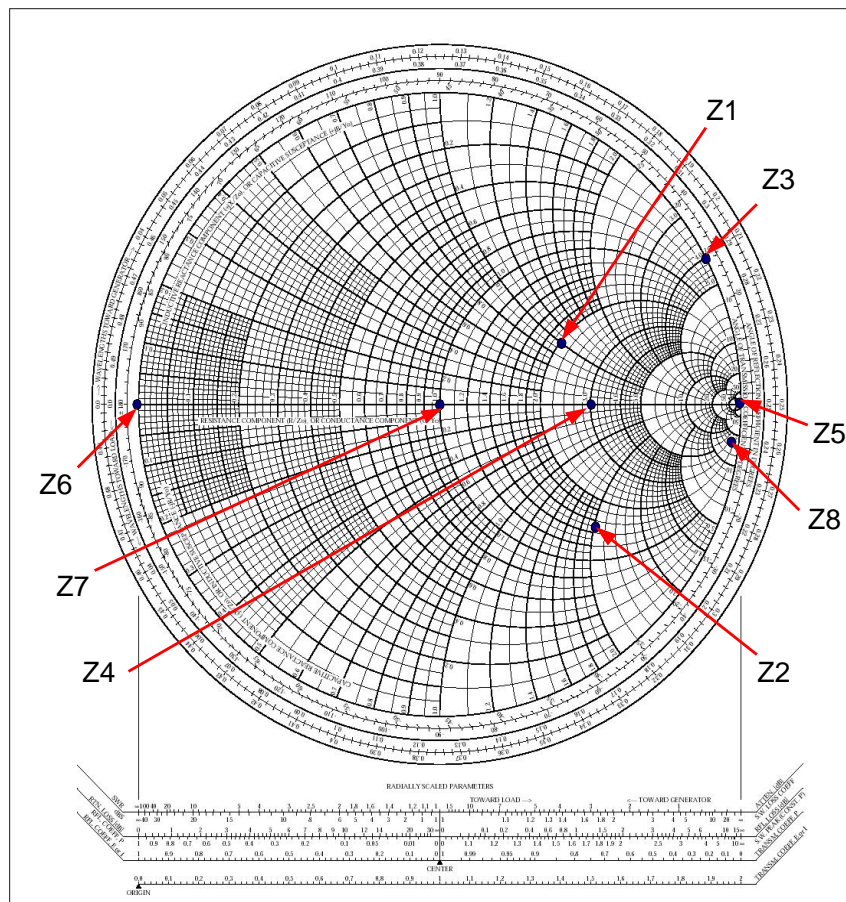


Figure 6: Points plotted on the Smith chart for Example 1.

The Smith chart is constructed by considering impedance (resistance and reactance). It can be used to analyse these parameters in both the series and parallel worlds. Adding elements in a series is straightforward. New elements can be added and their effects determined by simply moving along the circle to their respective values. However, summing elements in parallel is another matter, where admittances should be added.

We know that, by definition, $Y = 1/Z$ and $Z = 1/Y$. The admittance is expressed in mhos or Λ^{-1} or alternatively in Siemens or S. Also, as Z is complex, Y must also be complex. Therefore

$$Y = G + jB, \quad (10)$$

where G is called the conductance and B the susceptance of the element. When working with admittance, the first thing that we must do is normalize $y = Y/Y_0$. This results in $y = g + jb = 1/z$. So, what happens to the reflection coefficient? We note that

$$\Gamma = \frac{z-1}{z+1} = \frac{(z-1)/z}{(z+1)/z} = \frac{1-y}{1+y} \quad (11)$$

Thus, for a specific normalized impedance, say $z_1 = 1.7 + j0.6$, we can find the corresponding reflection coefficient as $\Gamma_1 = 0.33 \angle 28^\circ$. From (11), it then follows that the reflection coefficient for a normalized admittance of $y_2 = 1.7 + j0.6$ will be $\Gamma_2 = -\Gamma_1 = 0.33 \angle (28^\circ + 180^\circ)$.

This also implies that for a specific normalized impedance z , we can find $y = 1/z$ by rotating z

through an angle of 180° around the centre of the Smith chart on a constant radius (see Figure 7).

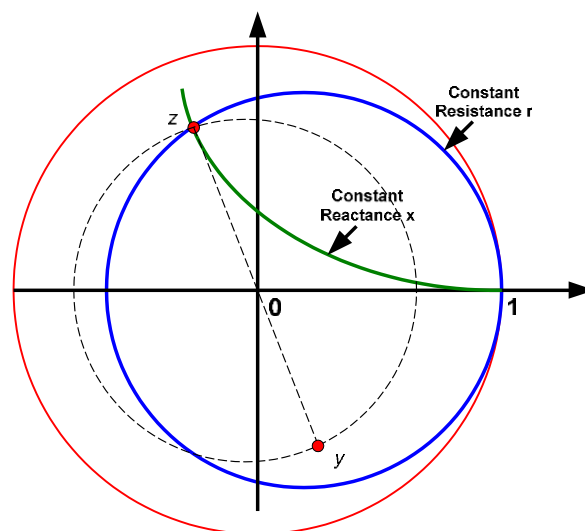


Figure 7: Results of the 180° rotation

Note that while z and $y = 1/z$ represent the same component, the new point has a different position on the Smith chart and a different reflection value. This is due to the fact that the plot for z is an impedance plot, but for y it is an admittance plot. When solving problems where elements in series and in parallel are mixed together, we can use the same Smith chart by simply performing rotations where conversions from z to y or y to z are required.

2 Smith Charts and transmission line circuits

So far we have based the construction of the Smith chart on the definition of the voltage reflection coefficient at the load. The question is: what happens when we connect the load to a length of transmission line as in Figure 8.

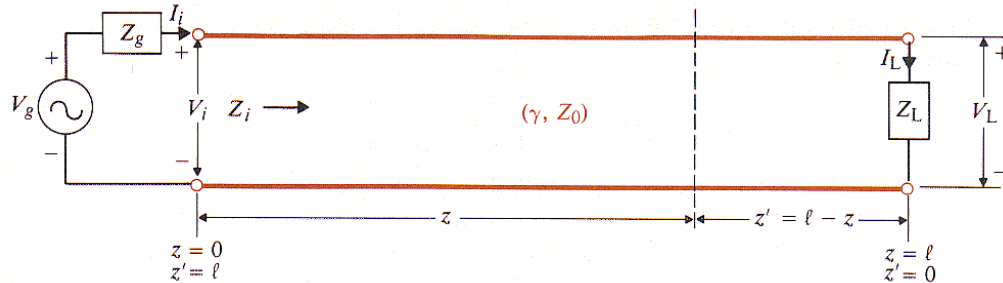


Figure 8: Finite transmission line terminated with load impedance Z_L .

On a lossless transmission line with $k = \beta$, the input impedance at a distance z' from the load is given by

$$Z_i = \frac{V(z')}{I(z')} = Z_0 \frac{1 + \Gamma_L e^{-j2\beta z'}}{1 - \Gamma_L e^{-j2\beta z'}} \quad (12)$$

The normalised impedance is then

$$z_i = \frac{Z_i}{Z_0} = \frac{1 + \Gamma_L e^{-j2\beta z'}}{1 - \Gamma_L e^{-j2\beta z'}} = \frac{1 + \Gamma_i}{1 - \Gamma_i} \quad (13)$$

Consequently, the reflection coefficient seen looking into the lossless transmission line of length z' is given by

$$\Gamma_i = \Gamma_L e^{-j2\beta z'} = \Gamma_L e^{j\theta} e^{-j2\beta z'} \quad (14)$$

This implies that as we move along the transmission line towards the generator, the magnitude of the reflection coefficient does not change; the angle only changes from a value of θ at the load to a value of $(\theta - 2\beta z')$ at a distance z' from the load. On the Smith chart,

we are therefore rotating on a constant $|\Gamma|$ circle. One full rotation around the Smith chart

requires that $2\beta z' = 2\pi$, so that $z' = \pi/\beta = \lambda/2$ where λ is the wavelength on the transmission line.

Two additional scales in z'/λ are usually provided along the perimeter of the $|\Gamma| = 1$ circle

for easy reading of the phase change $2\beta z'$ due to a change in line length z' . The scale is marked in "wavelengths towards generator" in the clockwise direction (increasing z') and "wavelengths towards load" in the counter-clockwise direction (decreasing z'). Figure 9

shows a typical commercially available Smith chart.

Each $|\Gamma|$ -circle intersects the real axis at two points. Refer to Figure 5. We designate the point

on the positive real axis as P_M and on the negative real axis as P_m . Since $x = 0$ along the real axis, both these points represent situations of a purely resistive input impedance, $Z_i = R_i + j0$. Obviously, $R_i > Z_0$ at P_M where $r > 1$, and $R_i < Z_0$ at P_m where $r < 1$. At the

point P_M we find that $Z_i = R_i = S Z_0$, while $Z_i = R_i = Z_0 / S$ at P_m . The point P_M on an

impedance chart corresponds to the positions of a voltage maximum (and current minimum) on the transmission line, while P_m represents a voltage minimum (and current maximum). Given an arbitrary normalised impedance z , the value of the r -circle passing through the point P_M is numerically equal to the standing wave ratio. For the example, if $z = 1.7 + j0.6$, we find

that $r = 2$ at P_M , so that $S = r = 2$.

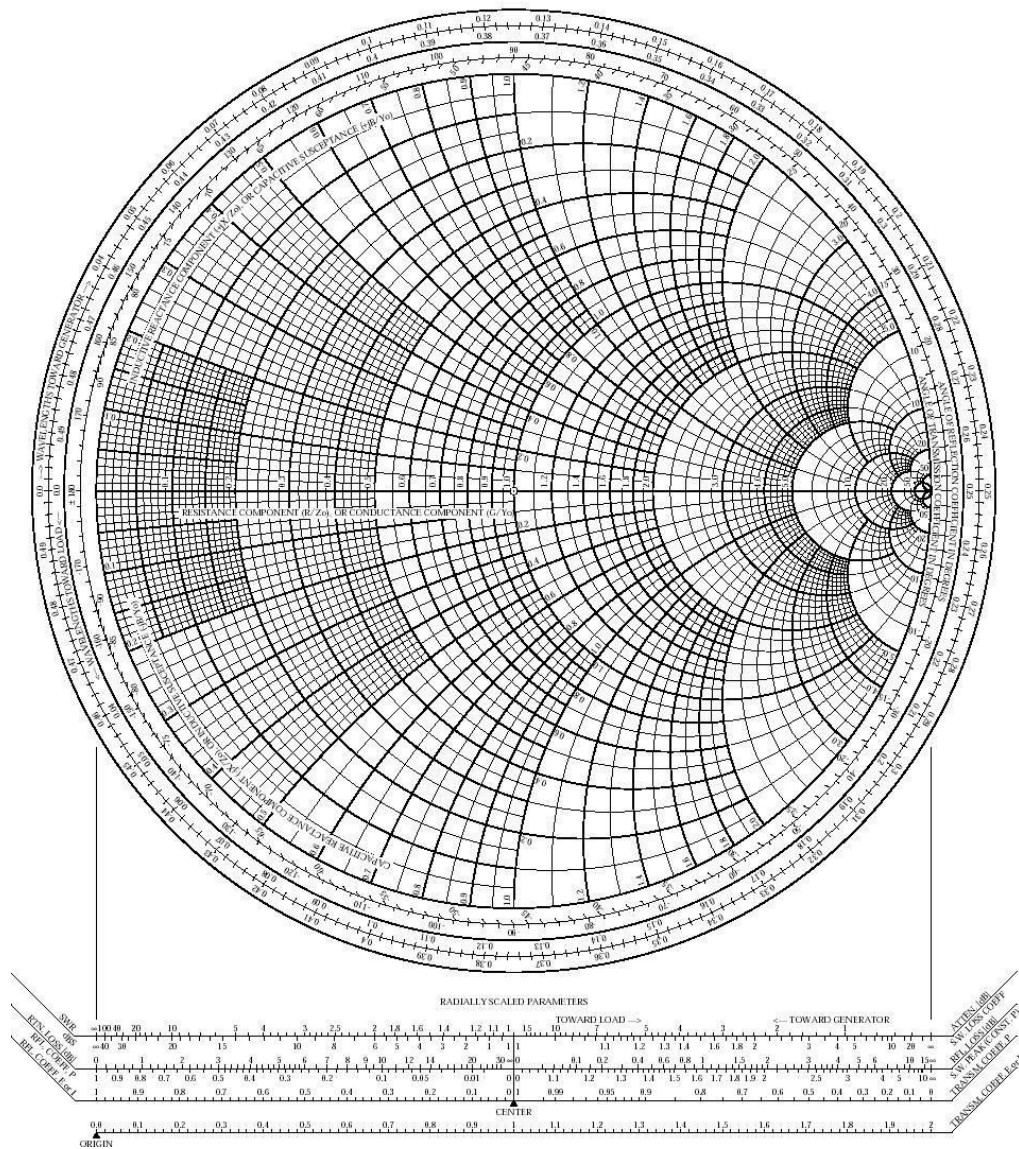


Figure 9: The Smith chart.

Example 2:

Use the Smith chart to find the impedance of a short-circuited section of a lossless $50 \, \Omega$ axial transmission line that is 100 mm long. The transmission line has a dielectric of relative permittivity $\epsilon_r = 9$ between the inner and outer conductor, and the frequency under consideration is 100 MHz.

For the transmission line, we find that

$$\beta = \omega \sqrt{\epsilon_r} = 6.2875 \text{ rad/m}$$

$\lambda = 2\pi/\beta = 0.9993 \approx 1$ m. The transmission line of length $z' = 100$ mm is therefore

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 $z'/\lambda = 0.1$ wavelengths long.

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- Since $z_L = 0$, enter the Smith chart at a point P_{sc} .
- Move along the perimeter of the chart ($\Gamma = 1$) by 0.1 “wavelengths towards the generator” in a clockwise direction to point P_1 .
- At P_1 , read $r = 0$ and $x \approx 0.725$, or $z_i = j0.725$. Then $Z_i = j0.725 \times 50 = j36.3 \Omega$.

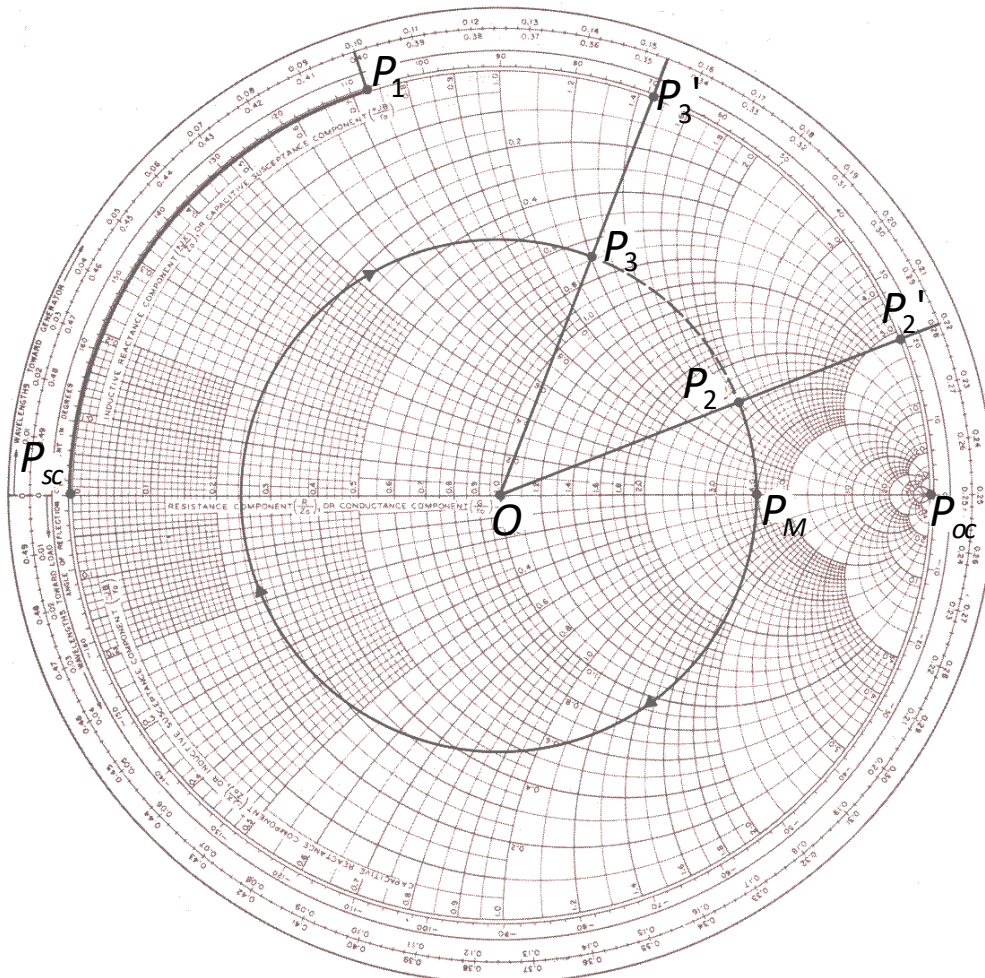


Figure 10: Smith chart calculations for Example 2 and Example 3.

Example 3: A lossless transmission line of length 0.434λ and characteristic impedance 100Ω is terminated in an impedance $260 + j180 \Omega$. Find the voltage reflection coefficient, the standing-wave ratio, the input impedance, and the location of a voltage maximum on the line.

Given $z' = 0.434\lambda$, $Z_0 = 100 \Omega$ and $Z_L = 260 + j180 \Omega$. Then

- Enter the Smith chart at $z_L = Z_L / Z_0 = 2.6 + j1.8$ shown as point P_2 in Figure 10.
- With the centre at the origin, draw a circle of radius $\overline{OP_2} = \Gamma_L = 0.6$.
- Draw the straight line OP_2 and extend it to P_2' on the periphery. Read 0.220 on

"wavelengths towards generator" scale. The phase angle θ of the load reflection may either be read directly from the Smith chart as 21° on the "Angle of Reflection Coefficient" scale. Therefore $\Gamma_L = 0.6 e^{j21\pi/180} = 0.6 e^{j0.12\pi}$.

- The $\Gamma = 0.6$ circle intersects the positive real axis OP_{sc} at $r = S = 4$. Therefore the

- To find the input impedance, move P_2' at 0.220 by a total of 0.434 “wavelengths toward the generator” first to 0.500 (same as 0.000) and then further to 0.434–(0.500–0.220)=0.154 to P_3' .

- Join O and P_3 by a straight line which intersects the $|\Gamma| = 0.6$ circle at P_3 . Here $r = 0.69$ and $x = 1.2$, or $z_i = 0.69 + j1.2$. Then $Z_i = (0.69 + j1.2) \times 100 = 69 + j120 \Omega$.
- In going from P_2 to P_3 , the $|\Gamma| = 0.6$ circle intersects the positive real axis at P_M where there is a voltage maximum. Thus the voltage maximum appears at $0.250 - 0.220 = 0.030$ wavelengths from the load.

3 Transmission line impedance matching.

Transmission lines are often used for the transmission of power and information. For RF power transmission, it is highly desirable that as much power as possible is transmitted from the generator to the load and that as little power as possible is lost on the line itself. This will require that the load be matched to the characteristic impedance of the line, so that the standing wave ratio on the line is as close to unity as possible. For information transmission it is essential that the lines be matched, because mismatched loads and junctions will result in echoes that distort the information-carrying signal.

Impedance matching by quarter-wave transformer

For a lossless transmission line of length l , characteristic impedance of terminated in a load impedance Z_L , the input impedance is given by

$$Z_0 = R_0 \quad \text{and}$$

$$\begin{aligned} Z &= R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l} \\ &= R_0 \frac{Z_L + jR_0 \tan(2\pi l / \lambda)}{R_0 + jZ_L \tan(2\pi l / \lambda)} \end{aligned} \quad (15)$$

If the transmission line has a length of $l = \lambda / 4$, this reduces to

$$\begin{aligned} Z &= R_0 \frac{Z_L + jR_0 \tan(\pi / 2)}{R_0 + jZ_L \tan(\pi / 2)} \\ &= R_0 \frac{Z_L / \tan(\pi / 2) + jR_0}{R_0 / \tan(\pi / 2) + jZ_L} \\ &= R_0 \frac{0 + jR_0}{0 + jZ_L} \\ &= \frac{(R_0)^2}{Z_L} \end{aligned} \quad (16)$$

This presents us with a simple way of matching a resistive load $Z_L = R_L$ to a real-valued

input impedance $Z_i = R_i$: insert a quarter-wave transformer with characteristic impedance of

R_0 . From (16), we have $R_i = (R_0)^2 / R_L$, or

$$R_0 = \sqrt{R_i R_L} . \quad (17)$$

Note that the length of the transmission line has to be chosen to be equal to a quarter of a transmission line wavelength at the frequency where matching is desired. This matching method is therefore frequency sensitive, since the transmission line section will no longer be a quarter of a wavelength long at other frequencies. Also note that since the load is usually matched to a purely real impedance $Z_i = R_i$, this method of impedance matching can only be applied to resistive loads $Z_L = R_L$, and is not useful for matching complex load impedances to a lossless (or low-loss) transmission line.

Example 4

A signal generator has an internal impedance of 50Ω . It needs to feed equal power through a lossless 50Ω transmission line with a phase velocity of $0.5c$ to two separate resistive loads ϕ

64 Ω and 25 Ω at a frequency of 10 MHz. Quarter-wave transformers are used to match the loads to the 50 Ω line, as shown in Figure 11.

- Determine the required characteristic impedances and physical lengths of the quarter-wavelength lines.
- Find the standing-wave ratios on the matching line sections.

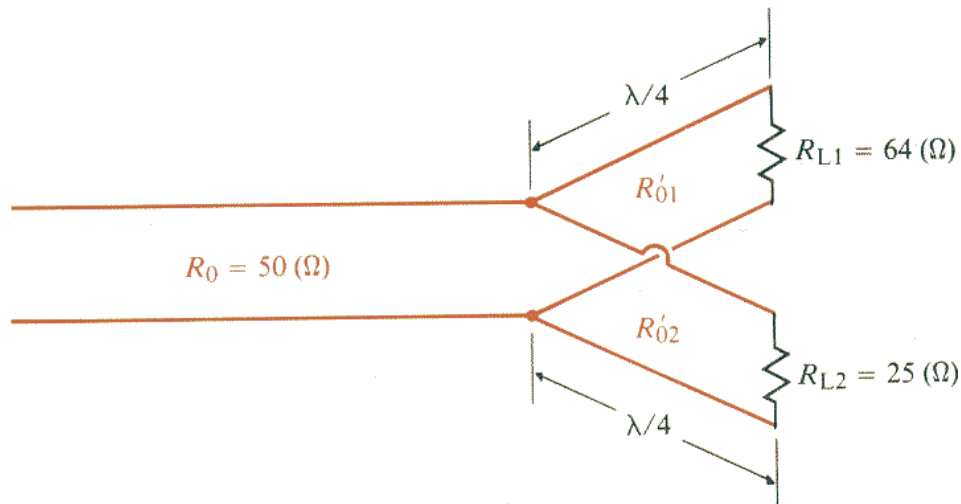


Figure 11: Impedance matching by quarter-wave transformers (Example 4).

- To feed equal power to the two loads, the input resistance at the junction with the main line looking toward each load must be

$$R_{i1} = 2R_0 = 100 \Omega \quad \text{and} \quad R_{i2} = 2R_0 = 100 \Omega$$

Therefore

$$R_{01} = \sqrt{R_{i1} R_{L1}} = 80 \Omega$$

$$R_{02} = \sqrt{R_{i2} R_{L2}} = 50 \Omega$$

Assume that the matching sections use the same dielectric as the main line. We know that

$$u_p = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0 \epsilon_0 \epsilon_r}} = \frac{c}{\epsilon_r}$$

We can therefore deduce that it uses a dielectric with a relative permittivity of $\epsilon_r = 4$.

$$\lambda = \frac{u_p}{f} = \frac{2\pi}{k} = 15 \text{ m.}$$

The length of each transmission line section is therefore $l = \lambda / 4 = 3.75 \text{ m}$.

- Under matched conditions, there are no standing waves on the main transmission line, i.e.

$S = 1$. The standing wave ratios on the two matching line sections are as follows:

Matching section No. 1:

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$$\Gamma_{L1} = \frac{R_{L1} - R_0}{R_{L1} + R_0} = \frac{64 - 80}{64 + 80} = -0.11$$

$$S = \frac{1 + |\Gamma_{L1}|}{1 - |\Gamma_{L1}|} = \frac{1 + 0.11}{1 - 0.11} = 1.25$$

Matching section No. 2:

$$\Gamma_{L2} = \frac{R_{L2} - R_0}{R_{L2} + R_0} = \frac{25 - 50}{25 + 50} = -0.33$$

$$S = \frac{1 + |\Gamma_{L2}|}{1 - |\Gamma_{L2}|} = \frac{1 + 0.33}{1 - 0.33} = 1.99$$

Single stub matching

In matching of impedances, we are only allowed to use reactive components (i.e. equivalent to inductors and capacitors – no resistors). Recall that for short-circuited and open-circuited lossless transmission line sections of length l , the input impedance was given by

$$Z_{i,s} = jZ_0 \tan \beta l = jZ_0 \tan(2\pi l / \lambda), \quad (18)$$

and

$$Z_{i,o} = -jZ_0 \cot \beta l = -jZ_0 \cot(2\pi l / \lambda), \quad (19)$$

where $Z_0 = R_0$ is purely real. The impedances in (18) and (19) are purely reactive

(imaginary), and therefore these transmission line sections act as inductors or capacitors, depending on the line length. We are going to make use of these elements (called transmission line *stubs*) to design matching circuits. In practice, it is more convenient to use short-circuited stubs. Short-circuited stubs are usually used in preference to open-circuited stubs because an infinite terminating impedance is more difficult to realise than a zero terminating impedance. Radiation from the open end of a stub makes it appear longer than it is, and compensation for these effects makes the use of open-circuited stubs more cumbersome. A short-circuited stub of an adjustable length is much easier to construct than an open-circuited stub.

It is also more common to connect these stubs in parallel with the main line. For parallel connections, it is convenient to use admittances rather than impedances. In these cases, we use the Smith chart as an admittance chart to design the matching networks.

A single-stub matching circuit is depicted in Figure 12. Note that the short-circuited stub is connected in parallel with the main line. In order to match the complex load impedance Z_L to the characteristic impedance of the lossless main line, $Z_0 = R_0$, we need to determine the lengths d and l .

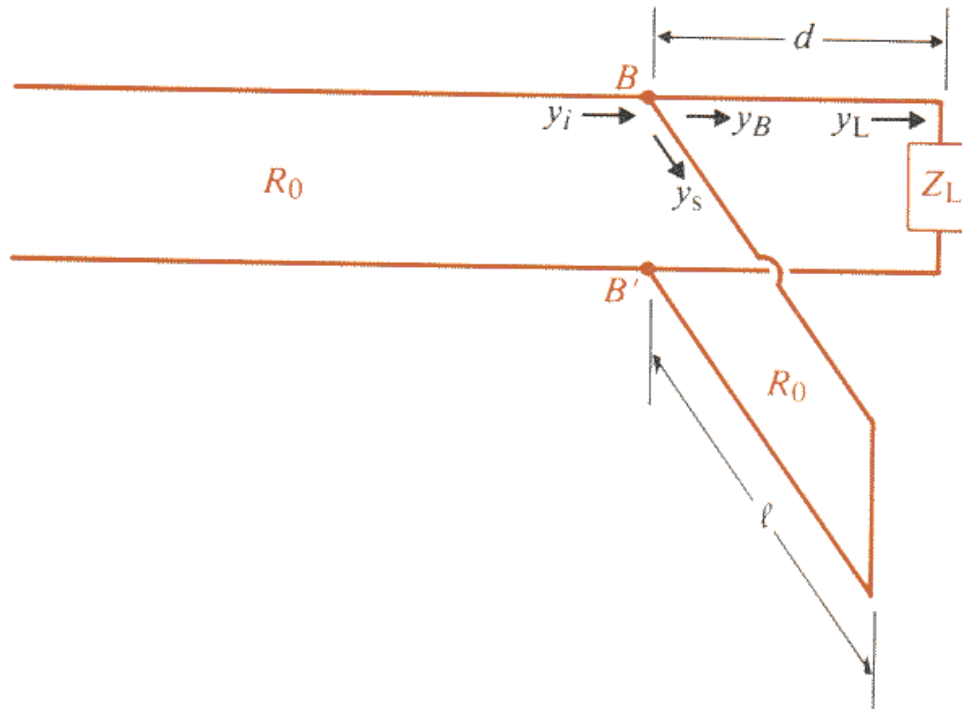


Figure 12: Impedance matching by single stub method.

For the transmission line to be matched at the point $B - B'$, the basic requirement is

$$\begin{aligned} Y_i &= Y_B + Y_s \\ &= Y_0 = \frac{1}{R_0} \end{aligned} \quad (20)$$

In terms of normalised admittances, (23) becomes

$$y_i = y_B + y_s = 1. \quad (21)$$

where $y_B = g_B + jb_B = Y_B / Y_0$ for the load section and $y_s = Y_s / Y_0$ for the short-circuited stub. Note that $y_s = -j \cot(2\pi l / \lambda)$ is purely imaginary. It can therefore only contribute to the imaginary part of y_i . The position of $B - B'$ (or, in other words, the length d) must be chosen such that $g_B = 1$, i.e.

$$y_B = 1 + jb_B. \quad (22)$$

Next, the length l is chosen such that

$$y_s = -jb_B, \quad (23)$$

which yields $y_i = y_B + y_s = (1 + jb_B) + (-jb_B) = 1$. The circuit is therefore matched at

$B - B'$, and at any point left of $B - B'$ as well.

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If we use the Smith chart, we would rotate on a Γ_L circle in a clockwise direction (towards the generator) when transforming the normalised load admittance to the admittance y_B .

However, according to (23), y_B must also be located on the $g = 1$ circle.

The use of the Smith chart for the purpose of designing a single-stub matching network is best illustrated by means of an example.

Example 5: A $50\ \Omega$ transmission line is connected to a load impedance $Z_L = 35 - j37.5\ \Omega$. Find the position and length of a short-circuited stub required to match the load at a frequency

of 200 MHz. Assume that the transmission line is a co-axial line with a dielectric for which

$$\epsilon_r = 9.$$

Given $Z_0 = R_0 = 50 \Omega$ and $Z_L = 35 - j47.5 \Omega$. Therefore $z_L = Z_L / Z_0 = 0.7 - j0.95$.

- Enter the Smith chart at z_L shown as point P_1 in Figure 13.
- Draw a Γ circle centred at O with radius $\overline{OP_1}$.
- Draw a straight line from P_1 through O to point P_2 on the perimeter, intersecting the $\Gamma = 1$ circle at P_2 , which represents y_L . Note 0.109 at P_2 on the "wavelengths toward generator" scale.
- Note the two points of intersection of the Γ circle with the $g = 1$ circle:
 - At P_3 : $y_{B1} = 1 + j1.2 = 1 + jb_{B1}$
 - At P_4 : $y_{B2} = 1 - j1.2 = 1 - jb_{B2}$
- Solutions for the position of the stub:
 - For P_3 (from P_2 to P_3) $d_1 = (0.168 - 0.109)\lambda = 0.059\lambda$
 - For P_4 (from P_2 to P_4) $d_2 = (0.332 - 0.109)\lambda = 0.223\lambda$
- Solutions for the length of the short-circuited stub to provide $y_s = -jb_B$:
 - For P_3 (from P_{sc} on the extreme right of the admittance chart to P'_3 , which represents $y_s = -jb_{B1} = -j1.2$): $l_1 = (0.361 - 0.250)\lambda = 0.111\lambda$
 - For P_4 (from P_{sc} on the extreme right of the admittance chart to P'_4 , which represents $y_s = -jb_{B2} = j1.2$): $l_2 = (0.139 + 0.250)\lambda = 0.389\lambda$

To compute the physical lengths of the transmission line sections, we need to calculate the wavelength on the transmission line. Therefore

$$\lambda = \frac{u_p}{f} = \frac{1/\sqrt{\mu\epsilon}}{f} = \frac{c/\epsilon_r}{f} \approx 0.5 \text{ m}.$$

Thus:

$d_1 = 0.059\lambda = 29.5 \text{ mm}$	$l_1 = 0.111\lambda = 55.5 \text{ mm}$
$d_2 = 0.223\lambda = 111.5 \text{ mm}$	$l_2 = 0.389\lambda = 194.5 \text{ mm}$

Note that either of these two sets of solutions would match the load. In fact, there is a whole range of possible solutions. For example, when calculating d_1 , instead of going straight from P_2 to P_3 , we could have started at P_2 , rotated clockwise around the Smith chart n times

(representing an additional length of $n\lambda / 2$) and continued on to P_3 , yielding

$d_1 = 0.059\lambda + n\lambda / 2$, $n = 0, 1, 2, \dots$. The same argument applies for d_2 , l_1 and l_2 .

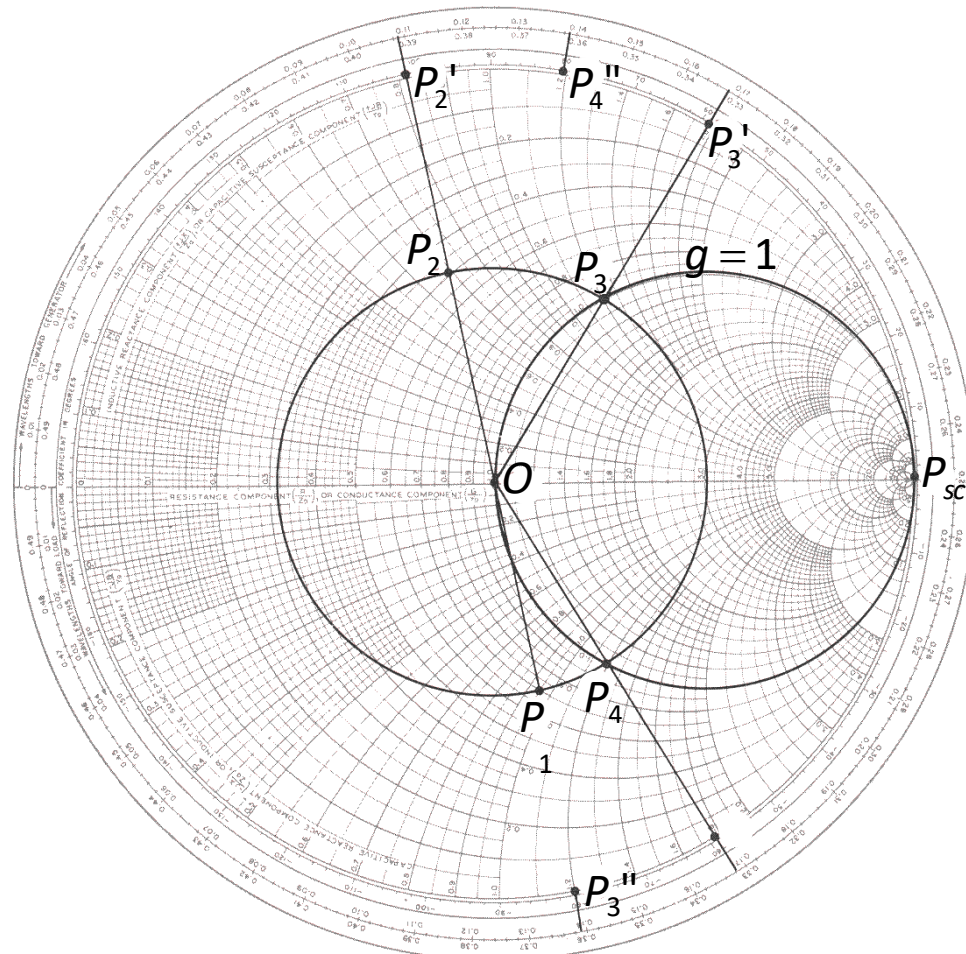


Figure 13: Single-stub matching on an admittance chart (Example 5).

